Surfaces with triple points

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Abstract

In this paper we compute upper bounds for the number of ordinary triple points on a hypersurface in $\mathbb{P}^3$ and give a complete classification for degree six (degree four or less is trivial, and five is elementary). But the real purpose is to point out the intricate geometry of examples with many triple points, and how it fits with the general classification of surfaces.

Introduction

The problem of finding the maximal number of simple singularities on projective hypersurfaces has attracted a lot of attention in the last twenty years. In this paper we study surfaces with a simple kind of non-simple singularities, namely ordinary triple points. In contrast to the case of simple surfaces singularities (classified by DuVal as those which 'do not affect the conditions of adjunction' [DV]) the invariants of the surface and its type in the classification of surfaces may change. Normal surfaces with higher singularities provide interesting examples of surfaces found, so to speak, in our back-yard.

We warm up by looking at quintics with isolated triple points. Their analysis is very elementary, but yields examples which nicely illustrate many aspects of the general theory of surfaces. Quintics with many triple points were to our knowledge first studied by Gallarati [G].

For higher degree surfaces the location of the triple points will matter very much, and the problem of finding the maximal number becomes very difficult. We derive several bounds on the number of triple points. We found an example of a septic with a high number (16) of triple points, which is one short of our upper bound. For sextics we give a classification, which takes up the main part of this paper.

Our research started out as search for sextics with many triple points. In particular, a sextic with 11 triple points would be very interesting. In fact, given

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the right configuration lying on a quadric, it would furnish a birational Abelian surface. However, we could not find such a surface; the putative example turned out to be a triple cover of a quadric. Then it was easily remarked that 11 triple points are *a priori* impossible. Instead we found many examples with 9 triple points and general arguments allowing to rule out possibilities. The successful construction of an example raises the question how special the construction is: can it be generalised? This can be decided by infinitesimal methods. Let Σ_ν^d be the stratum of surfaces with ν ordinary triple points in the parameter space of all surfaces of degree d. A lower bound for the dimension of Σ_ν^d in the point representing an explicit example X is the number of moduli in the construction plus 15 from coordinate transformations — the stabiliser of the point configurations being discrete for large ν. An upper bound is given by the dimension of the Zariski tangent space. If these dimensions are equal we know that the stratum is smooth of the given dimension in that point, and the construction gives the general element of the component of Σ_ν^d. To compute the Zariski tangent space we have to determine which polynomials of degree d induce equisingular deformations of the singular points, which for each specific example can easily be computed with *Macaulay* [B–M].

The clue to classifying sextics with many triple points is the study of exceptional curves of the first kind on the minimal resolution. It turns out that only a few different cases can occur. For nine triple points, we find three families of $K3$ surfaces (theorem 4.13) and two families of properly elliptic surfaces (theorem 4.14). In the $K3$ cases we find for each possible configuration of the nine points a pencil of sextics of the form $\alpha g + \beta q^3$, where q defines the (unique) quadric through the nine points, and g is a reducible sextic. In the other two cases we find even a net of sextics, again containing q^3 and reducible sextics. Constructing reducible sextics with triple points is not so difficult.

Regarding the existence of a sextic $\{g = 0\}$ with 10 triple points we first observe that the pencil $\alpha g + \beta q^3$, where $\{q = 0\}$ passes through nine of them, falls into one of our five families of sextics with nine triple points. Assuming that an element of such a family has a tenth triple point gives conditions on the coefficients. The resulting equations are much to difficult to solve. By imposing extra symmetry we have been able to reduce the number of variables and equations, while keeping at least a one-parameter family of solutions.

This paper is organised as follows. First we study quintics with triple points. The next section describes the birational invariants of our surfaces. In the third section several bounds for the number of triple points on a surface of degree d are given. Then we identify the exceptional curves. The next section is the main part of this paper. First we study the exceptional curves on a sextic with triple points and find only a few different cases, depending on the geometric genus (corollary 4.4). Based on the geometry of the exceptional curves, we give an overview over sextic surfaces with $0, \ldots, 10$ ordinary triple points. For convenience, a list of sextics with triple points and their invariants is given (theorem 4.18). In the last

section a septic with 16 ordinary triple points is constructed.

1 Quintics

A smooth quintic in $\mathbb{P}^3$ is one of the simplest examples of a surface of general type. Its Chern invariants are given by $c_1^2 = 5$, $c_2 = 55$ and thus $\chi = 4$. Furthermore it has the Hodge invariants $p_g = 4$, $q = 0$ and $h^{1,1} = 45$ (we give general formulas in the next section). The adjoint system consists of planes, thus is 4-dimensional (illustrating $p_g = 4$).

If nodes are allowed, or more generally rational double points, nothing happens to the invariants, as the corresponding resolutions are smooth deformations of smooth quintics. It is however an interesting question to try and decide what 'bouquets' of rational double points can be imposed. The maximal number of nodes is 31 (see below), and one may impose five E_8 singularities ($w^5 = F(x, y, z)$ where F is a plane quintic with five cusps) which is somewhat short of the maximal number $44 = h^{1,1} - 1$ (counted with multiplicity according to Milnor number) theoretically possible.

If other types of singularities are considered, more interesting things happen. Invariants change, and also the type of the surfaces. There exist classifications of quintics with isolated singularities (at least for those of general type), and interesting such examples occur for $\tilde{E}_8$ singularities (normal form: $z^2 = y^3 + \lambda x^2 y^2 + x^6$). We will however restrict ourselves only to ordinary triple points (locally $f(x, y, z) = 0$ with f a smooth plane cubic). It is easy to see (cf. Section 2), that a triple point decreases c_1^2 by three and c_2 by nine (and thus χ by one). Furthermore the adjoint system consists of planes passing through the triple points. We can thus establish the following table, where ν denotes the number of triple points.

ν	c_1^2	c_2	χ	p_q	q	Type of surface
0	5	55	5	4	0	general type
1	2	46	4	3	0	general type
2	-1	37	3	2	0	elliptic blown up once
3	-4	28	2	1	0	$K3$ blown up four times
4	-7	19	1	0	0	rational
5	-10	20	0	0	1	ruled over elliptic curve

The most interesting thing about this table is the geometry of the special examples, which nicely illustrates many aspects of the general theory of surfaces. We should also note that the constructions of surfaces are elementary, as we can impose the triple points generically, and then simply solve linear equations in the coefficients. We observe that the line joining any two triple points lies by Bezout on the surface, and furthermore is exceptional. The latter is true for any conic passing through three triple points *and* lying on the quintic.

Let us now comment upon our small menagerie of surfaces, found, so to speak in our back-yard.

$\nu = 1$: This is a double octic, the double cover of $\mathbb{P}^2$ effected by projection from the triple point. (In fact all the quintics with triple points, can be considered as double octics, with one less triple point.) Note that not all double octics are of this type.

$\nu = 2$: An elliptic surface blown up. The elliptic fibration is given by the planes through the line joining the two triple points. The intersection consists of the line and a quartic with two double points at its intersection with the line. The resolution of such plane quartics (a resolution effected by the desingularisation of the triple points) is clearly elliptic. The canonical divisor will coincide with the elliptic fibration (plus the exceptional divisor). The elliptic curves which arise from desingularisation, will be bi-sections of the elliptic fibration.

$\nu = 3$: A $K3$ surface blown up four times. The canonical divisor is given by the plane through the three triple points. The intersection of the quintic with that plane is given by the three lines of the corresponding triangle, and the residual conic, passing through them all. If you make the resolved surface minimal, you get a $K3$ surface with three elliptic curves E_i all passing through a common point p and any two E_i, E_j $(1 \leq i \neq j \leq 3)$ also intersecting in another point p_k (i, j, k distinct integers). Each of those elliptic curves will give rise to elliptic fibrations, which are amusing to identify. All the $K3$ surfaces will have Picard number at least three, and the configuration of the elliptic curves will give a common sublattice to them all. Conversely starting with a $K3$ surface with such a sublattice of its Picard group, we choose a point p and elliptic curves E_i in each of the corresponding pencils, passing through p. Those will define intersection points $p_k = E_i \cap E_j$ (i, j, k distinct). Blowing up the points to exceptional divisors F_k and G (the latter corresponding to the blow up of p) we can in fact write down the divisor H of degree 5, effecting the birational embedding. Namely

$$H = E_1 + E_2 + E_3 - F_1 - F_2 - F_3 - 2G \, .$$

It is straightforward to check that $H^2 = 5$, $H \cdot E_i = 0$ and $H \cdot F_i = 1$ while $H \cdot G = 2$. Independent examples of such $K3$ surfaces are furnished by quartics with three lines, each of which gives rise to an elliptic fibration. It could be a mildly challenging exercise for the reader to see how such a quartic can be transformed into a quintic with three triple points. It is also amusing to count parameters: there are 56 quintic monomials, 10 conditions for a triple point at a fixed location, and 6-dimensional family of projective linear transformations, fixing three points. This makes 19. On

the other hand $K3$ surfaces with Picard number at least three, make up a 17-dimensional family, and we add 2 for the position of the point.

$\nu = 4$: A rational surface blown up many times (15 or 16 times). You expect an infinite number of exceptional curves of the first kind. Ten of those are obvious, given by the six edges of the tetrahedron spanned by the triple points, and the four residual conics corresponding to each face. It is a challenge to find others.

$\nu = 5$: A ruled surface over an elliptic curve, blown up ten times. The twenty exceptional divisors, which come in pairs, are obvious. Each pair of triple points determine an exceptional line, and dually the three remaining triple points a residual exceptional conic. This also gives a clue to the ruling. Through six points one may always find a unique twisted cubic. For each point on the surface, consider the twisted cubic through it and the five triple points. By Bezout this curve has to lie on the surface. The resolved elliptic curves will be sections. By blowing down the ten lines (or the ten conics) we get a minimal ruled surface. This will turn out to be the one coming from the stable rank-two bundle on an elliptic curve.

Note that six or more triple points is an impossibility. Such a putative surface would necessarily have $\chi < 0$ and hence be ruled over a curve of genus $g > 1$. In such surfaces there is no space for elliptic curves, as they can neither surject on to the base, nor squeeze into the fibres.

Quintics with at least four triple points can be simplified using a birational transformation, known as *reciprocal transformation* [SR, VIII § 4]. The ordinary plane Cremona transformation using the linear system of conics through three points can be described in suitable coordinates by the formula $(x:y:z) \mapsto (1/x:1/y:1/z)$. This formula generalises to higher dimensions. In particular, the space transformation

$$(x:y:z:w) \mapsto \left(\frac{1}{x}:\frac{1}{y}:\frac{1}{z}:\frac{1}{w}\right)$$

simultaneously blows up the vertices and blows down the faces of the coordinate tetrahedron. The vertices are called *fundamental points* of the reciprocal transformation. Let $Y \subset \mathbb{P}^3$ by a surface of degree d not containing any of the coordinate planes. Let $m_1, \ldots, m_4$ be the multiplicities of Y in the fundamental points. Then the image Y' of Y is a surface of degree $3d - m_1 - \cdots - m_4$. In many cases Y' will be singular in the fundamental points with singularities obtained from contracting the intersection curves of Y with the coordinate planes.

For a quintic X the following happens: taking four triple points as fundamental points we transform the surface into one of degree $3 \cdot 5 - 4 \cdot 3 = 3$. Conversely, given four points on a cubic surface we obtain a surface of degree $3 \cdot 3 - 4 = 5$

with four new singularities, which are ordinary triple points if the tetrahedron (spanned by the four points) cuts out smooth curves. This argument shows that five triple points are maximal, and realisable by starting with a cubic cone. The construction also allows you to find the blown up rational quintic with four triple points. As we all know the cubic can be thought of $\mathbb{P}^2$ blown up six times. The four vertices of the tetrahedron will provide four more blow ups, and finally the six edges of the tetrahedron, each intersect the cubic in a residual point, each of which is blown up. This makes a total of sixteen.

It is amusing to continue. Setting H to be the hyperplane section of the cubic (thus $H^2 = 3$) and letting E_i denote the exceptional curves associated to the four vertices, and E_{ij} the residual intersections associated to the six edges, we can write down the linear system on the cubic, which gives the quintic, as

$$3H - 2\sum_i E_i - \sum_{ij} E_{ij}\ .$$

This linear system blows down the four elliptic curves

$$H - E_i - E_j - E_k - E_{ij} - E_{ik} - E_{jk}$$

each of which has self-intersection $3 - 6 = -3$. Furthermore the six exceptional curves E_{ij} are mapped onto lines, namely the edges of the tetrahedron spanned by the triple points, while the exceptional curves E_i are mapped onto residual conics. Each of the 27 lines of the cubic, which does not pass through a vertex, maps onto a twisted cubic passing simply through all four triple points. As there are exceptional curves of arbitrary high degree on the cubic blow-up, we see that on a quintic there may be exceptional curves of arbitrary high degree.

The above analysis is very elementary, and parts of it has not too surprisingly already appeared in the literature [G]. It can be generalised in a number of different ways. One, which we have already mentioned, is to consider more general singularities (in this way interesting surfaces of general type can be constructed [Y]), the other is to consider triple points on surfaces of higher degree.

2 Invariants

A surface singularity $P \in X$ is an *ordinary triple point of* X if there exist local coordinates x, y and z centred at P such that X is given by the equation

$$x^3 + y^3 + z^3 + \lambda xyz = 0$$

for a $\lambda \in \mathbb{C}$ with $\lambda^3 \neq -27$. Such a triple point is also called a singularity of type $\tilde{E}_6$ (resp. P_8, $T_{3,3,3}$). The minimal resolution of an ordinary triple point is given as follows: let $\pi_P \colon \tilde{X} \to X$ be the blowup of X in P and let E_P be the exceptional divisor. This is a smooth elliptic curve, given in suitable homogeneous

coordinates $(x:y:z)$ by the equation $x^3+y^3+z^3+\lambda xyz=0$. In particular $\tilde{X}$ is smooth in every point of E_P and the self intersection of E_P on $\tilde{X}$ is -3.

Now let $X \subset \mathbb{P}^3$ be a projective surface of degree d with ν isolated triple points. Let $\mathcal{S}=\{P_1,\dots,P_\nu\}=X_{sing}$ be the singular locus of X and let $\tilde{X}$ be the blow-up of X in $\mathcal{S}$; it is a smooth model of X, however not minimal in general. We will denote the minimal model of X by $\overline{X}$. Moreover let $E=\sum_{i=1}^{\nu} E_i$ be the sum of all exceptional divisors.

There are basically two types of invariants of $\tilde{X}$. To start with, there are invariants of local nature which take into account the number, but not the position of the triple points. This are the Chern numbers $c_1(\tilde{X})^2$, $c_2(\tilde{X})$ and the holomorphic Euler characteristic $\chi(\mathcal{O}_{\tilde{X}})$. Second, there are invariants which are also influenced by the position of the triple points. Amongst them are the geometrical genus $p_g(\tilde{X})$, the irregularity $q(\tilde{X})$, the Betti numbers $b_i(\tilde{X})$, the Hodge numbers $h^{p,q}(\tilde{X})$ and the Kodaira dimension $\kappa(\tilde{X})$. We are going to compare these invariants with the invariants of a smooth hypersurface X_s of degree d.

The canonical class and $\boldsymbol{c_1(\tilde{X})^2}$: we have $K_{\tilde{X}} \sim_{lin} \pi^* K_{X_s} - E$. This follows from the adjunction formula: we know that $E_i \cdot (K_{\tilde{X}} + E_i) = 0$, so if $K_{\tilde{X}} \sim_{lin} \pi^* K_X - \alpha E$, then $E_i \cdot (K_{\tilde{X}} + E_i) = E_i \cdot (\pi^* K_{X_s} - \alpha E + E_i) = 3(\alpha - 1)$. As ${K_X}^2 = {K_{X_s}}^2$ we find

$$c_1(\tilde{X})^2 = {K_{\tilde{X}}}^2 = {K_{X_s}}^2 - 3\nu. \tag{1}$$

Every triple point diminishes $c_1(\tilde{X})^2$ by three.

The Euler number $\boldsymbol{e(\tilde{X}) = c_2(\tilde{X})}$: the drop in the Euler number is purely local and can be computed by means of topological considerations from the Milnor number $\mu(\tilde{E}_6)=8$ and the Euler numbers $e(E_i)=e(torus)=0$ and $e(point)=1$ as follows:

$$\begin{aligned} e(\tilde{X}) &= e(X_s) + \nu\left(-e(point) + e(E_i) - \mu(\tilde{E}_6)\right) \\ &= e(X_s) - 9\nu. \end{aligned}$$

So every triple point diminishes the Euler number by nine.

The holomorphic Euler characteristic $\boldsymbol{\chi(\mathcal{O}_{\tilde{X}})}$: the Noether formula on $\tilde{X}$ says $\chi(\mathcal{O}_{\tilde{X}}) = \left(c_1(\tilde{X})^2 + c_2(\tilde{X})\right)/12$, hence

$$\chi(\mathcal{O}_{\tilde{X}}) = \chi(\mathcal{O}_{X_s}) - \nu.$$

Every triple point diminishes the holomorphic Euler characteristic by one.

A quick (cheating) way of computing the above 'drop' in invariants exploits the local character of the Chern invariants. So consider a smooth cubic, whose invariants are $c_1^2=3$ and $c_2=9$. A cubic with a triple-point is a cone over an

elliptic curve, the resolution is a ruled surface over an elliptic curve, and hence $c_1^2 = c_2 = 0$ giving indeed the 'drops' three and nine, deduced above.

Now we come to the other invariants. The adjoint linear system on X is cut out by those surfaces of degree $d-4$ which pass through every point of $\mathcal{S}$. Intuitively speaking, every triple point in general position puts a linear condition on the adjoint linear system of X, so it will diminish the geometric genus $p_g(\tilde{X})$ by one (if not already zero). Let $\alpha \in \mathbb{N}_0$ be the discrepancy defined by

$$p_g(\tilde{X}) = p_g(X_s) - \nu + \alpha.$$

As a consequence

$$q(\tilde{X}) = q(X_s) + \alpha.$$

Both $\tilde{X}$ and X_s are Kähler, so we have the Hodge decompositions of $H^i(\tilde{X}, \mathbb{Z}) \otimes \mathbb{C} \cong H^i(\tilde{X}, \mathbb{C})$ and $H^i(X_s, \mathbb{Z}) \otimes \mathbb{C} \cong H^i(X_s, \mathbb{C})$. From the equalities $h^{p,q} = h^{q,p} = h^{4-p,q}$ and $b_0(\tilde{X}) = b_0(X_s) = 1$ one easily computes

$$b_2(\tilde{X}) = b_2(X_s) - 9\nu + 4\alpha,$$
$$h^{1,1}(\tilde{X}) = h^{1,1}(X_s) - 7\nu + 2\alpha.$$

The other Betti numbers and Hodge numbers do not give more information: $b_3(\tilde{X}) = b_1(\tilde{X}) = 2q(\tilde{X})$, $h^{1,0}(\tilde{X}) = h^{0,1}(\tilde{X}) = h^{2,1}(\tilde{X}) = h^{1,2}(\tilde{X}) = q(\tilde{X})$, $h^{2,0}(\tilde{X}) = h^{0,2}(\tilde{X}) = p_g(\tilde{X})$ and $h^{0,0}(\tilde{X}) = h^{2,2}(\tilde{X}) = 1$.

We list the invariants in terms of d, ν and α in the following table.

	X_s	$\tilde{X}$
c_1^2	$d(d-4)^2$	$d(d-4)^2 - 3\nu$
c_2	$d(d^2-4d+6)$	$d(d^2-4d+6) - 9\nu$
χ	$d(d^2-6d+11)/6$	$d(d^2-6d+11)/6 - \nu$
p_g	$\binom{d-1}{3}$	$\binom{d-1}{3} - \nu + \alpha$
q	0	α
b_2	d^3-4d^2+6d-2	$d^3-4d^2+6d-2-9\nu+4\alpha$
$h^{1,1}$	$d(2d^2-6d+7)/3$	$d(2d^2-6d+7)/3 - 7\nu + 2\alpha$

Table 1: The invariants of X_s and $\tilde{X}$

The Kodaira dimension $\kappa(\tilde{X})$ measures the growth of the plurigenera $P_n(\tilde{X}) = h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(nK_{\tilde{X}}))$ as n grows. In general we have [BPV, ch. I, thm. 7.2]

$$\kappa(\tilde{X}) \begin{cases} \geq 0 & \text{if } p_g(\tilde{X}) \geq 1 \text{ and} \\ \geq 1 & \text{if } p_g(\tilde{X}) \geq 2. \end{cases}$$

For surfaces of general type $P_2(\tilde{X}) = {K_{\tilde{X}}}^2 + \chi(\mathcal{O}_{\tilde{X}}) + \epsilon$, where ϵ is the number of exceptional divisors. Generally we have $P_n(\tilde{X}) = \frac{1}{2}n(n-1)({K_{\tilde{X}}}^2 + \epsilon) + \chi(\mathcal{O}_{\tilde{X}})$.

3 Bounds for the number of triple points

The surface X can have only a finite number of ordinary triple points, the maximal number depending on its degree d. Let $\mu_3(d)$ be the maximal number of ordinary triple points of a degree d surface. We immediately find

$$\mu_3(1) = \mu_3(2) = 0, \quad \mu_3(3) = \mu_3(4) = 1 \quad \text{and} \quad \mu_3(5) = 5.$$

The only cubic surface with an ordinary triple point is the cone over a plane smooth elliptic curve. A quartic surface with two triple points is necessarily singular along the line joining these two points. As we have seen in the preceding section, for quintics the result is due to Gallarati [G].

The position of the triple points cannot be too special, as also the maximal number of triple points of X on a given curve or surface is bounded. Let $C \subset \mathbb{P}^3$ be a curve of degree c and $V \subset \mathbb{P}^3$ a surface of degree v.

Lemma 3.1

1) *C contains at most $c(d-1)/2$ triple points of X (with multiplicity).*

2) *If V and X do not have a common component, then V contains at most $vd(d-1)/6$ triple points of X (with multiplicity).*

Proof: Consider the linear system $\mathcal{L}_p$ of polar surfaces of X, i.e. the linear system generated by the partial derivatives of the degree d polynomial defining X. Then $\mathcal{S}$ is exactly the base locus of $\mathcal{L}_p$ and the general member $X_p \in \mathcal{L}_p$ is a degree $d-1$ surface which is smooth except ordinary double points in the triple points of X. So X_p does not contain a component of C. In every triple point $P \in \mathcal{S}$ the intersection multiplicity of C and X_p in P is $\operatorname{mult}_P(C, X_p) \geq 2$ and thus $2\nu \leq C \cdot X_p = c(d-1)$. This proves 1).

The surfaces V, X and X_p intersect in a finite number of points. In every triple point $P \in \mathcal{S}$ the intersection multiplicity of V, X and X_p in P is $\operatorname{mult}_P(V, X, X_p) \geq 6$. Hence $6\nu \leq V \cdot X \cdot X_p = vd(d-1)$ and 2) holds. $\square$

We will now discuss three bounds for $\mu_3(d)$ with $d \geq 6$: the polar bound, the Miyaoka bound and the spectrum bound.

The polar bound: Suppose that $p_g(\tilde{X}) \geq 1$. Taking a general adjoint surface $V = K_p$ we find using lemma 3.1 2)

$$\nu \leq \frac{1}{6} d(d-1)(d-4).$$

The condition $p_g(\tilde{X}) \geq 1$ is satisfied for $d > 6$. This can be seen as follows. Substitute $\alpha = p_g(\tilde{X}) - \binom{d-1}{3} + \nu$, then the inequalities $1 + \nu \leq h^{1,1}(\tilde{X})$ and $q(\tilde{X}) \geq 0$ imply

$$p_g(\tilde{X}) \geq \frac{1}{24}(d-1)\left(2d^2 - 16d + 21\right).$$

So for $d \geq 7$ we have even $p_g(\tilde{X}) \geq 2$. The bound $\nu \leq \frac{1}{6}d(d-1)(d-4)$, which we call polar bound, even holds for $d=6$ in case $p_g(\tilde{X}) = 0$. Then $b_2(\tilde{X}) = h^{1,1}(\tilde{X})$ and the equation $1+\nu \leq h^{1,1}(\tilde{X})$ gives

$$\nu \leq \left\lfloor \frac{1}{18}(d-1)(d^2+d-3) \right\rfloor = 10.$$

We get the following table.

d	5	6	7	8	9	10	11	12
$\nu \leq$	6	10	21	37	60	90	128	176

Table 2: The polar bound

The Miyaoka bound: Miyaoka's famous bound [M] applies only to quotient singularities of surfaces with nonnegative Kodaira dimension. However there is a generalisation by Wall [W, cor. 2] which also applies to log-canonical singularities on surfaces with nonnegative Kodaira dimension. Applied to triple points, we get the bound

$$\nu \leq \frac{2}{27}d(d-1)^2.$$

As $p_g(\tilde{X}) \geq 2$ for $d = \deg(\tilde{X}) \geq 7$ this bound holds for every surface with triple points of degree ≥ 7 and we get the following table.

d	7	8	9	10	11	12
$\nu \leq$	18	29	42	60	81	107

Table 3: The Miyaoka bound

The spectrum bound [AGV, sect. 14.3.2] uses the semicontinuity of the spectrum of a singularity. Let $f\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be the germ of an isolated hypersurface singularity with Milnor number μ. Then the characteristic polynomial of the monodromy has μ eigenvalues which are roots of unity, and the Mixed Hodge Structure on the cohomology of the Milnor fibre gives a way to take logarithms. The precise definitions are not important for us now. The spectrum is easy to compute for a function of the form $f = x_0^{a_0} + \ldots + x_n^{a_n}$: then the spectrum is the set of rational numbers (with multiplicity) of the form $i_0/a_0 + \ldots + i_n/a_n$ with the i_j running from 1 to $a_j - 1$. Specifically we can take $a_i = d$ for all i, and as the spectrum is invariant under μ-constant deformations, we have it now for any homogeneous isolated singularity. A projective hypersurface with isolated singularities has a smooth hyperplane section, and the affine complement of the section is a small deformation of the affine cone over the hyperplane section, so a

homogeneous isolated singularity. The important property of the spectrum is its semicontinuity, in the sense that for every open interval of length 1 the number of spectral numbers in it of the singularity in the special fibre is at least the sum of the spectral numbers in the same interval of all singularities in the general fibre of a 1-parameter deformation (of negative degree). We consider the spectrum as a divisor on $\mathbb{Q}$. With this notation the spectrum for $d = 5$ is

$$1\left(\tfrac{3}{5}\right) + 3\left(\tfrac{4}{5}\right) + 6\left(\tfrac{5}{5}\right) + 10\left(\tfrac{6}{5}\right) + 12\left(\tfrac{7}{5}\right) + 12\left(\tfrac{8}{5}\right) + 10\left(\tfrac{9}{5}\right) + 6\left(\tfrac{10}{5}\right) + 3\left(\tfrac{11}{5}\right) + 1\left(\tfrac{12}{5}\right)$$

The spectrum of an ordinary double point is $\left(\frac{3}{2}\right)$. As the open interval $\left(\frac{3}{5}, \frac{8}{5}\right)$ contains 31 spectral numbers, a quintic surface can contain at most 31 nodes.

The spectrum for an $\tilde{E}_6$ is

$$1\left(\tfrac{3}{5}\right) + 3\left(\tfrac{4}{5}\right) + 3\left(\tfrac{5}{5}\right) + 1\left(\tfrac{6}{5}\right).$$

The open interval $\left(\frac{4}{5}, \frac{9}{5}\right)$ contains 40 spectrum numbers of the quintic, and seven of $\tilde{E}_6$, so $\lfloor \frac{40}{7} \rfloor = 5$ is the spectrum bound. Analogous computations for higher degree give the following table.

d	5	6	7	8	9	10	11	12
$\nu \leq$	5	11	17	29	45	60	84	114

Table 4: The spectrum bound

Putting all bounds together we arrive at the

Proposition 3.2 *Let $X \subset \mathbb{P}^3$ be a surface of degree $d \geq 3$ with ν ordinary triple points as its only singularities. Then ν is bounded as given by the following table.*

d	3	4	5	6	7	8	9	10	11	12
$\nu \leq$	1	1	5	10	17	29	42	60	81	107

We want to classify surfaces in $\mathbb{P}^3$ with only ordinary triple points. In contrast to surfaces with only ordinary double points, the class of a surface can change if more triple points come into play. The more triple points, the less nef $K_{\tilde{X}}$ will become and this makes the Kodaira dimension eventually drop. The cases $d \leq 4$ being obvious, we can state the

Proposition 3.3 *If $d \geq 7$, then $\tilde{X}$ is minimal. If $d = 6$, then the smooth rational (-1)-curves on $\tilde{X}$ come from rational curves C of degree $c \geq 2$ on X through $2c + 1$ triple points (with multiplicity), while for $d = 5$ they come from curves of degree $c \geq 1$ on X through $c + 1$ triple points.*

Proof: Let $C \subset X$ be a rational curve of degree c such that $\tilde{C} \subset \tilde{X}$ is a smooth rational (-1)-curve. Then $-2 = \deg K_{\tilde{C}} = (K_{\tilde{X}} + \tilde{C})|_{\tilde{C}} = K_{\tilde{X}} \cdot \tilde{C} + \tilde{C}^2 = c(d-4) - \operatorname{mult}(C, \mathcal{S}) - 1$. Hence

$$\operatorname{mult}(C, \mathcal{S}) = c(d-4) + 1.$$

Applying lemma 3.1 1) we find that

$$c(d-4) + 1 = \operatorname{mult}(C, \mathcal{S}) \leq \frac{1}{2} c(d-1)$$

and consequently $c(d-7) \leq -2$. This implies $d \leq 6$ and $c \geq 2$ for $d = 6$. $\square$

Corollary 3.4 *If $d \geq 7$, then X is a minimal surface of general type.*

Proof: For $d \geq 7$ $\tilde{X}$ is minimal by proposition 3.3. $K_{\tilde{X}}$ is effective, so by the Enriques-Kodaira classification we just have to show $c_1(\tilde{X})^2 > 0$. But $c_1(\tilde{X})^2 = d(d-4)^2 - 3\nu$, so $c_1(\tilde{X})^2 \leq 0$ iff

$$\nu \geq \frac{1}{3} d(d-4)^2.$$

Playing this inequality against the Miyaoka bound gives a contradiction. $\square$

4 Sextics

4.1 Exceptional curves

We intend to study the (-1)-curves on $\tilde{X}$. The amazing thing is that there are severe restrictions. The consequences of lemma 3.1 and proposition 3.3 are as follows.

- At most two triple points lie on a line $L \subset \mathbb{P}^3$.
- At most five triple points lie in a plane $H \subset \mathbb{P}^3$ and if so, then $H \cdot X = 3C$ for a smooth conic C, and $\tilde{C} \subset \tilde{X}$ is a smooth rational (-1)-curve.

The conic through five triple points gives the simplest example of a (-1)-curve (this really does occur, see section 4.2 for explicit examples). Such a conic will be called a (-1)-conic; by abuse of notation we also call the curve $C \subset X$ a (-1)-curve, if $\tilde{C} \subset \tilde{X}$ is an exceptional curve of the first kind. The next possible candidates would be a twisted cubic curve through seven triple points and a rational quartic curve through nine triple points. Surprisingly the twisted cubic is impossible.

Proposition 4.1 *At most six triple points lie on a cubic curve $D \subset X$.*

Proof: By lemma 3.1 1), a cubic curve contains at most seven triple points. So assume that $\text{mult}(D, \mathcal{S}) = 7$. Then D cannot be a plane curve and D cannot split in three lines. So either $D = C + L$ for a nondegenerate conic C and a line L with $\text{mult}(C, \mathcal{S}) = 5$ and $\text{mult}(L, \mathcal{S}) = 2$ or D is a twisted cubic curve.

In the first case let H be the plane containing C. Then $H \cdot X = 3C$, so L and C meet in one point $P \in X \setminus \mathcal{S}$. But then $L \cdot X \geq 7$, so $L \subset X$. Hence $L \subset T_{P,X} = H$, contradiction.

In the second case let N be the net of quadrics with the twisted cubic D as its base locus. So the general (smooth) quadric $Q \in N$ intersects X in S and a residual curve D_Q of type $(4, 5)$ with double points in $D \cap \mathcal{S}$. But on Q we have $D \cdot D_Q = (2, 1).(4, 5) = 14$. Hence $D \cap D_Q = D \cap \mathcal{S}$ for the general $Q \in N$. But for every $P \in D \setminus \mathcal{S}$ there exists a pencil of quadrics $N_P \subset N$ having contact to X at P. This implies that $P \in D_Q$ for all $Q \in N_P$. Now two things can happen. Either $N_P \subset N$ moves if we move P on $D \setminus \mathcal{S}$ or N_P is constant. If N_P moves it will sweep out a Zariski open subset of $N \simeq \mathbb{P}^2$. Then $D \cap D_Q = D \cap \mathcal{S}$ cannot hold for the general element of N. So N_P is constant and for every $Q \in N_P$, X has contact to Q along D. But then for two elements $Q \neq Q' \in N_P$ one has $Q \cdot Q' = 2D + D'$ for some curve D', contradiction. $\square$

Corollary 4.2 *There is no (-1)-curve of degree three on X.*

In the case $p_g(\tilde{X}) \geq 1$ we find further restrictions for the number of (-1)-curves and their degrees. First we show that every (-1)-curve is irreducible. Whenever the canonical divisor $K_{\tilde{X}}$ is effective, any exceptional divisor E is automatically a component, as $K_{\tilde{X}} \cdot E = -1 < 0$. Therefore E comes from a rational curve on X of degree c (by Proposition 3.3 through $2c + 1$ triple points), which is contained in the base locus of the system of quadrics through the triple points. This is the adjoint system; we will call every quadric in it a *canonical surface*.

Proposition 4.3 *Let C be an irreducible (-1)-curve on X, let K be a canonical divisor (of degree 12) and C' the residual curve of C in K. Then the strict transform $\tilde{C} \subset \tilde{X}$ of C is disjoint from the strict transform $\tilde{C}'$ of C'.*

Proof: First suppose that C is a conic. Then the residual curve C' has degree 10, and no component of it lies in the plane through C. Therefore the intersection multiplicity $C \cdot C'$ is at most 10. As C' has multiplicity 2 in each of the five triple points on C, the intersection multiplicity is exactly 10 and $\tilde{C}$ is disjoint from $\tilde{C}'$.

If $\deg C \geq 4$, then C lies on an irreducible quadric Q. We shall show that $\tilde{C}$ is disjoint from $\tilde{C}'$ on the blow up of Q in the points $P \in \mathcal{S}$. We first suppose that Q is smooth. Then C is a curve of type (a, b) with arithmetic genus $p_a = (a-1)(b-1)$, and C' has type $(6-a, 6-b)$, so $C \cdot C' = 6(a+b) - 2ab$. Suppose that C has multiplicity 3 in τ points $P \in \mathcal{S}$, multiplicity 2 in δ points, and passes simply though σ points. Then $3\tau + 2\delta + \sigma = 2(a + b) + 1$. As C is rational we

have that $3\tau + \delta \leq p_a$. This gives $\delta + \sigma \geq 3(a+b) - ab$. The multiplicity of $C \cup C'$ is three in each point $P \in \mathcal{S}$, so $C \cdot C' \geq 2\delta + 2\sigma$. Therefore we find

$$\delta + \sigma \geq 3(a+b) - ab \geq \delta + \sigma \, .$$

So C intersects C' only in points $P \in \mathcal{S}$ and the blow up of these points separates both curves.

The case that Q is a quadric cone with vertex outside $\mathcal{S}$ is handled in the same way. As C is smooth outside $\mathcal{S}$ and there does not intersect C' we conclude that C does not pass through the vertex.

Finally we investigate the case that the vertex of Q is a point $P \in \mathcal{S}$. Then $K = C \cup C'$ has multiplicity 6 in P. Let $\overline{Q}$ be the blow up of Q in the point P. Its Picard group is generated by E and f, with $E^2 = -1$, $E \cdot f = 1$ and $f^2 = 0$; we have $K_{\overline{Q}} \sim -2E - 4f$. The strict transform of K is a curve of type $3E + 12f$. Let C have multiplicity m in P, then its strict transform $\overline{C}$ is a curve of type $aE + (2a+m)f$, with $p_a(\overline{C}) = (a-1)(a+m-1)$. We have $\overline{C}' \sim (3-a)E + (12-2a-m)f$, so $\overline{C} \cdot \overline{C}' = 12a + 3m - 2a(a+m)$. Let $\overline{C}$ have τ triple, δ double and σ simple points in $\mathcal{S} \setminus \{P\}$. Then $3\tau + 2\delta + \sigma = 4a + m + 1$, $3\tau + \delta \leq p_a(\overline{C})$ and $\overline{C} \cdot \overline{C}' \geq 2\delta + 2\sigma$. Therefore

$$\delta + \sigma \geq 6a + 2m - a(a+m) \geq \delta + \sigma + \frac{1}{2}m \, .$$

We conclude that $m = 0$, so C does not pass through P, and that $\tilde{C}$ is disjoint from $\tilde{C}'$. $\square$

Corollary 4.4 *If $p_g(\tilde{X}) = 1$, then the degree of every (-1)-curve is one of $\{2, 4, 5, 6, 7, 8\}$. Moreover there are at most 6 such disjoint curves. If $p_g(\tilde{X}) = 2$ there are at most two (-1)-curves of degree 2 or 4 and if $p_g(\tilde{X}) \geq 3$, there is at most one (-1)-curve of degree 2.*

Proof: In first case all (-1)-curves are contained in the unique canonical curve of degree 12. Moreover $c_1(\tilde{X})^2 = -3$ and $p_g(\tilde{X}) \neq 0$, so $\tilde{X}$ has at least three (-1)-curves, which are disjoint by proposition 4.3. In the second case the base locus of the adjoint system is a curve of degree ≤ 4. In the last case the base locus of of the adjoint system is a curve of degree ≤ 3. Now the proposition follows because there are no (-1)-curves of degree 1 or 3. $\square$

We can now determine $p_g(\tilde{X})$ for $\nu = 9, 10$.

Corollary 4.5 *If $\nu = 9$, then $p_g(\tilde{X}) = 1$.*

Proof: Let $\nu = 9$, then $p_g(\tilde{X}) \geq 1$. If $p_g(\tilde{X}) \geq 2$ corollary 4.4 implies that $\tilde{X}$ has at most two (-1)-curves, which contradicts $c_1(\tilde{X})^2 = -3$ and $p_g(\tilde{X}) \neq 0$. $\square$

Corollary 4.6 *If $\nu = 10$, then $p_g(\tilde{X}) = 0$.*

Proof: Let $\nu = 10$, then $c_1(\tilde{X})^2 = -6$. So if $p_g(\tilde{X}) > 0$, then $\tilde{X}$ contains at least six (-1)-curves. Then $p_g(\tilde{X}) = 1$ by corollary 4.4 . The only possibility for six (-1)-curves is six conics $C_1, \ldots, C_6$ which make up $K_{\tilde{X}}$. Blowing down the six conics gives a minimal surface $\overline{X}$ with $c_1(\overline{X})^2 = 0$, $c_2(\overline{X}) = 12$ and $K_{\overline{X}} = \mathcal{O}_{\overline{X}}$. But there is no such surface in the Enriques-Kodaira classification. $\square$

In fact, an X with 10 triple points and $p_g(\tilde{X}) = 1$ would have an equation of the form $h_1 \cdots h_6 + q^3$ where the h_i define planes. Six planes intersect in 20 triple points. It is possible to choose 10 of them under the condition that no three lie on a line, but those points never lie on a quadric. We found our first example of a sextic with 9 triple points by taking $h_1 \cdots h_6 + q^3$ with q defining a quadric through 9 of the 10 points, chosen as required.

Now let $C \subset \tilde{X}$ be a rational quartic curve such that $\tilde{C} \subset \tilde{X}$ is a smooth rational (-1)-curve. C is contained in a smooth quadric surface and is either of type $(2,2)$ or $(1,3)$.

- If C is of type $(2,2)$, then C has one double point in a triple point of X. Moreover C passes simply through seven other triple points. C is the base locus of a pencil of quadrics whose general member is smooth.

- If C is of type $(1,3)$, then C is smooth and passes simply through nine triple points of X.

The case of a quartic (-1)-curve of type $(1,3)$ turns out to be impossible.

Lemma 4.7 *Every quartic (-1)-curve on X is of type $(2,2)$.*

Proof: Assume that C_1 is a rational quartic (-1)-curve on X of type $(1,3)$. Then C_1 is smooth and passes simply through nine triple points of X, so $\nu \in \{9, 10\}$. Moreover C_1 is contained in a unique smooth quadric $Q = \{q = 0\}$. We have two cases.

Case $\boldsymbol{\nu = 9}$: Then Q is the unique canonical surface (corollary 4.5). Every (-1)-curve is contained in the degree twelve curve $K = Q \cdot X$. No five triple points lie on a plane, so there exist no (-1)-conics. But $c_1(\tilde{X})^2 = -3$ and $p_g(\tilde{X}) = 1$, so there are at least three (-1)-curves. By corollary 4.4, the only possibility is three (-1)-curves: the curve C_1 of type $(1,3)$ and two other quartic (-1) curves C_2 and C_3 of types $(3,1)$ and $(2,2)$. Both C_1 and C_2 have multiplicity one in all points of $\mathcal{S}$, whereas C_3 misses one triple point. This contradicts $K = C_1 + C_2 + C_3$ and $\mathrm{mult}(K, P) = 3$ for all $P \in \mathcal{S}$.

Case $\boldsymbol{\nu = 10}$: By corollary 4.6 we have $p_g(\tilde{X}) = 0$, so Q passes exactly through nine triple points. If $X = \{f = 0\}$, then a general element of the pencil defined by $\alpha f + \beta q^3 = 0$ is a sextic with $\nu = 9$ ordinary triple points and C_1 as (-1)-curve. Hence we are done using the first case. $\square$

As a further consequence we get the useful

Corollary 4.8 *If C_1, $C_2 \subset X$ are two different (-1)-conics, then C_1 and C_2 meet in two distinct triple points of X.*

Proof: Let H_i be the plane containing C_i, $i = 1, 2$. Then $H_i \cdot X = 3C_i$, so $C_1 \cap C_2 \neq \emptyset$. Moreover H_i is a tangent plane to X at every point of C_i. Thus every point of $C_1 \cap C_2$ is a singular point, i.e. a triple point. If there is just one such point, then $\nu \geq 9$ and $p_g(\tilde{X}) = 11 - \nu$, which contradicts one of the corollaries 4.5 and 4.6. $\square$

Assume that X has $\nu = 9$ triple points $P_1, \ldots, P_9$. Let Q be the unique (irreducible) canonical quadric surface and let $K = Q \cdot X$ be the adjoint curve. The resolution $\tilde{X}$ has at least three disjoint (-1)-curves, which all are components of K. There are two main possibilities: either K is the union of all (-1)-curves or not. In the first case blowing them down gives a minimal surface $\overline{X}$ with $K_{\overline{X}} = \mathcal{O}_{\overline{X}}$, so $c_1(\overline{X})^2 = 0$ and there are exactly three (-1)-curves with degrees c_1, c_2 and c_3. It follows from corollary 4.4 that up to permutation

$$(c_1, c_2, c_3) \in \{(2,2,8), (2,4,6), (2,5,5), (4,4,4)\}.$$

In the second case we end up with an effective canonical divisor after blowing down. In this case up to permutation the possible degrees are

$$(2,2,2), (2,2,2,2), (2,2,2,2,2), (2,2,4), (2,2,2,4), (2,2,5), (2,2,2,5),$$
$$(2,2,6), (2,2,2,6), (2,2,7), (2,4,4), (2,4,5).$$

First we will rule out some cases. For the remaining cases, we will give explicit examples, when making a tour from zero to ten triple points, studying all possible cases.

Proposition 4.9 *If the (-1)-curves do not make up K, then there are exactly three with degrees $(2,2,2)$ or $(2,2,4)$.*

Proof: We exclude all other possibilities case by case.

Suppose first that there are three (-1)-conics. As there are only nine triple points the only possibility is that the three planes containing the conics have only one point in common and that each intersection line contains two triple points, while the remaining three points each lie in only one plane. Assume now that there is a fourth (-1)-conic. Its five triple points have to lie on the intersection triangle with the first three planes with at least two triple points in the vertices. This implies that there are three triple points on a line, thus excluding the cases $(2,2,2,2)$ and $(2,2,2,2,2)$. Also $(2,2,2,4)$ is not possible: a quartic (-1)-curve C passes through 8 triple points, and there is at least one plane H_i containing five of them, but $C \cdot H_i = 4$.

Case $(2,2,5)$: As $\mathrm{mult}(C_3, \mathcal{S}) = 11 = 9 + 2$ the curve C_3 has either one triple point or two double points in $\mathcal{S}$. But every irreducible degree five curve on Q

has arithmetic genus 0 or 2. Every double point drops the genus by at least one, every triple point by at least three (as C_3 does not pass through the vertex of Q if Q is singular). The only possibility is that C_3 has two double points. As C_3 is not a plane curve, it meets the plane of a (-1)-conic simply in the five triple points. But the two (-1)-conics contain together eight triple points, so C_3 can have at most one double point, contradiction. This excludes also the case $(2, 2, 2, 5)$.

The cases $(2, 2, 6)$, $(2, 2, 2, 6)$ and $(2, 2, 7)$ are similar.

Case $(2, 4, 4)$: Both C_2 and C_3 are of type $(2, 2)$ and have a singular point in $\mathcal{S}$ outside H_1. Counting intersection points of C_2 and C_3 gives a count ≥ 9 (with multiplicity). This contradicts $C_2 \cdot C_3 = 8$.

The case $(2, 4, 5)$ is similar. $\square$

We see that under the conditions of the proposition the degree of a (-1)-curve is always even. We shall show that this also holds if the degrees sum up to 12 by excluding the case $(2, 5, 5)$. It is possible to construct a reducible curve on Q consisting of a curve of type $(1, 1)$, $(2, 3)$ and $(3, 2)$ with the required intersection behaviour. So we need a different type of argument.

Proposition 4.10 *If the (-1)-curves make up K, then*

$$(c_1, c_2, c_3) \in \{(2, 2, 8), (2, 4, 6), (4, 4, 4)\}.$$

Proof: Let $(c_1, c_2, c_3) = (2, 5, 5)$, heading for a contradiction. Let $P_1, \ldots, P_5$ be the five triple points on the conic C_1. On $\tilde{X}$ we have $3C_1 \sim_{lin} H - E_1 - \ldots - E_5$, so

$$3(C_2 + C_3) \sim_{lin} 5H - 2(E_1 + \cdots + E_5) - 3(E_6 + \cdots + E_9).$$

Therefore there exists a degree five surface $Y = \{g = 0\}$ with multiplicity two at the points $P_1, \ldots, P_5$ and triple points $P_6, \ldots, P_9$ such that $f = hg - q^3$. Here $Q = \{q = 0\}$ is the canonical quadric and $H = \{h = 0\}$ is the plane containing $C_1 = \{q = h = 0\}$. As Y intersects Q in two irreducible curves of odd degree, Y is itself irreducible. A plane through three triple points intersects Y in the triangle formed by the triple points and a residual conic through them. If one of the five double points lies in the plane, the conic degenerates and has multiplicity two at the double point. But this would imply that three triple points of the original sextic lie on a line. Consider the reciprocal transformation centred in the triple points of Y. The image of Y will be an irreducible cubic surface Y' with five points of multiplicity two. Therefore Y' has nonisolated singularities: it has a double line. So Y itself has a double curve of degree at most three, passing through the five points $P_1, \ldots, P_5$. This means that the conic C_1 is a component of the double line, so the sextic surface X is singular along C_1. $\square$

4.2 Sextics with seven or less triple points

For all surfaces X with up to four triple points $\tilde{X}$ is minimal by proposition 3.3. No three triple points lie on a line, thus $p_g(\tilde{X}) = 10 - \nu$. There are no constraints on the position of the triple points except that no three are on a line, so we get an equation of X just by solving linear equations in the coefficients. If four triple points P_1, P_2, P_3, P_4 lie in a plane $H \subset \mathbb{P}^3$, then $X \cdot H$ is a degree six curve with four triple points and hence splits into three conics: $X \cdot H = C_1 + C_2 + C_3$. Then X has an equation of the form

$$hg + q_1 q_2 q_3 = 0$$

with $H = \{h = 0\}$ and $C_i = \{h = q_i = 0\}$, $i = 1, 2, 3$. Here g is a degree five form which vanishes in $P_1, \ldots, P_4$ to the second order. In any case, the base locus of the system of adjoint surfaces consists only of the triple points.

Imposing a fifth triple point P_5 opens the possibility of a (-1)-curve. This happens if and only if $P_1, \ldots, P_5$ lie on a plane H. Then $X \cdot H = 3C$ for a nondegenerate conic C and $\tilde{C} \subset \tilde{X}$ is the (-1)-curve. Every such surface has an equation of the form

$$hg + q^3 = 0$$

with $H = \{h = 0\}$ and $C = \{h = q = 0\}$. Here g is a degree five form vanishing doubly in $P_1, \ldots, P_5$. Then the base locus of the system of adjoint surfaces is exactly C. If the five triple points do not lie in a plane we find $\{P_1, \ldots, P_5\}$ as base locus. In any case $p_g(\tilde{X}) = 5$.

Let P_6 be another triple point. Since six triple points cannot lie on a conic the geometric genus will drop by one: $p_g(\tilde{X}) = 4$. We end up with the same cases as for $\nu = 5$ yielding as base locus $C \cup \{P_6\}$ resp. $\{P_1, \ldots, P_6\}$.

Examples of sextics with $\nu \leq 6$ triple points and base locus $\{P_1, \ldots, P_\nu\}$ of the adjoint system can be given as follows. Let $Q_i = \{q_i = 0\}$ be generators of the linear system of quadrics through $\{P_1, \ldots, P_\nu\}$, $i = 0, \ldots, 10 - \nu$. Then the general element of the linear system spanned by the mixed third powers of the q_i has only triple points in $P_1, \ldots, P_\nu$. For $\nu \leq 5$ every surface is of this form. For $\nu = 6$ the linear system of mixed third powers has dimension $\binom{6}{3} - 1 = 19$ while the system of all sextics with triple points has dimension $\binom{9}{6} - 1 - 6 \cdot 10 = 23$.

Now we go for a seventh triple point P_7. Again the geometric genus drops: $p_g(\tilde{X}) = 3$. The base locus cannot be a degree three curve by proposition 4.1. If $P_1, \ldots, P_5$ lie on a conic C, the base locus is $C \cup \{P_6, P_7\}$. If not, the net of quadrics defined by $P_1, \ldots, P_7$ has a zero-dimensional base of the form $\{P_1, \ldots, P_7, P\}$ for a eighth point $P \in \mathbb{P}^3$, which may be infinitely near to one of the points $P_1, \ldots, P_7$. Now the mixed third powers of the q_i have an additional singularity in P. They form a system of dimension 9, which is four less than the dimension of the system of all sextics with triple points. To find an equation of

such a surface it suffices to give one possibly reducible sextic not passing through P; the surface obtained by adding a general combination of third powers has then only 7 triple points. As such an extra sextic we can take the product $g_1 g_2$ of a cubic g_1 with nodes in $P_1, \ldots, P_4$ passing simply through P_5, P_6, P_7 and a cubic g_2 with nodes in P_5, P_6, P_7 passing simply through $P_1, \ldots, P_4$.

In all cases considered so far we end up with $c_1(\tilde{X})^2 > 0$ and $p_g(\tilde{X}) \geq 3$, so $\tilde{X}$ is a surface of general type.

4.3 Eight triple points

We distinguish the sextics with eight triple points by their geometric genus and their (-1)-curves.

We can always choose seven of the eight points so that no five lie on a conic. These seven triple points $P_1, \ldots, P_7$ (no three on a line, no five on a conic) determine a net of quadrics spanned by $Q_i = \{q_i = 0\}$, $i = 1, 2, 3$.

The case $p_g(\tilde{X}) = 3$**:** this means that the eighth triple point P_8 is the eighth base point of the net. For a general ternary cubic form f_3 the surface $X = \{f_3(q_1, q_2, q_3) = 0\}$ is a sextic with only triple points in $P_1, \ldots, P_8$. Here $p_g(\tilde{X}) = 3$, thus $q(\tilde{X}) = 1$ and $\tilde{X}$ is minimal because K is effective and has no fixed components, so $\tilde{X}$ is minimal properly elliptic. This elliptic surface is fibred over an elliptic curve, namely the plane elliptic curve given by $f_3 = 0$. The fibration is induced by the net, i.e. given by (q_1, q_2, q_3) and each fibre is the base locus of a pencil of quadrics, in a way the reader easily can work out. Note that the elliptic resolutions of the triple points are sections of this fibration, and hence all isomorphic to the base. (This can also be seen by noting that the linear parts of the q_i's at the basepoints are linearly independent.)

The case $p_g(\tilde{X}) = 2$**:** we assume that P_8 is not a base point of the net. Let the pencil of quadrics through $P_1, \ldots, P_8$ be spanned by Q_1 and Q_2. Let C be the base locus of the pencil. For the (-1)-curves we can have four different cases:

- one (-1)-curve of type $(2, 2)$,
- two (-1)-conics,
- one (-1)-conic and
- no (-1) curves at all.

One (-1)**-curve of type** $(2, 2)$**:** in this case the equation of X has a very special form.

Lemma 4.11 *If $\tilde{X}$ contains a quartic (-1)-curve C of type $(2, 2)$, then the equation of X has the form*

$$q_0 g + q^3 = 0$$

with $Q_0 = \{q_0 = 0\}$ a quadric cone through eight triple points with vertex in one of them and $Q = \{q = 0\}$ a smooth quadric through the eight triple points. The pencil of quadrics with base locus C is spanned by Q and Q_0. Moreover $Y = \{g = 0\}$ is a quartic surface passing through the vertex of Q_0 with seven double points in the other seven triple points.

Proof: Let $P_1 \in \mathcal{S}$ be the double point of C and let $P_2, \ldots, P_7$ be the other triple points on C. Let M be the pencil of quadrics with base locus C. The general element $Q \in M$ is smooth and intersects X in C and a residual curve C_Q of type $(4,4)$ passing simply through P_1 and doubly through $P_2, \ldots, P_7$. Since C and C_Q do not have a common component we have $C \cap C_Q = \{P_1, \ldots, P_8\}$ in view of $C \cdot C_Q = (2,2) \cdot (4,4) = 16$. Now fix a point $P \in C \setminus \mathcal{S}$. There exists a $Q_0 \in M$ which has contact to X at P. In particular $P \in C_{Q_0}$, implying that C and C_{Q_0} have a common component. Hence $X \cdot Q_0 = 2C + C'$ for a curve C' of type $(2,2)$. Thus $\mathrm{mult}(X \cdot Q_0, P_1) \geq 4$, so Q_0 is singular in P_1. Hence $\mathrm{mult}(C', P_1) \geq 2$ and $X \cdot Q_0 = 3C'$ by Bezout. In view of $C \subset Q_0$ the quadric Q_0 has to be a quadric cone with vertex P_1. Then X has an equation as demanded. $\square$

The condition that a quadric $\sum \lambda_i q_i$ in the net of quadrics through 7 points (in general position) is singular is that there exists a point P in which all derivatives of $\sum \lambda_i q_i$ vanish. Eliminating the λ_i gives that the maximal minors of

$$\begin{vmatrix} \frac{\partial q_1}{\partial x} & \frac{\partial q_1}{\partial y} & \frac{\partial q_1}{\partial z} & \frac{\partial q_1}{\partial w} \\ \frac{\partial q_2}{\partial x} & \frac{\partial q_2}{\partial y} & \frac{\partial q_2}{\partial z} & \frac{\partial q_2}{\partial w} \\ \frac{\partial q_3}{\partial x} & \frac{\partial q_3}{\partial y} & \frac{\partial q_3}{\partial z} & \frac{\partial q_3}{\partial w} \end{vmatrix}$$

have to vanish. This locus is known as the Steiner curve of the net. For Q we take a smooth quadric through the eight points. There exists a six parameter family of quartics through P_8 with only nodes in $P_1, \ldots, P_7$ (quadrics in the net give a five parameter family, but the general quartic is not of that form). Let $Y = \{g = 0\}$ be a general such quartic, then $X = \{q_0 g + q^3 = 0\}$ is a sextic with eight triple points and one (-1)-curve C of type $(2,2)$. Altogether this construction has 13 moduli: we find seven independent sextics with triple points in the given points, and the configuration of points has seven moduli. The minimal model $\overline{X}$ of X has $c_1\left(\overline{X}\right)^2 = 1$, so $\tilde{X}$ is of general type.

Such a surface has a characterisation as the minimal model of a double cover of F_2 (the resolution of a quadric cone), branched along a 5-section disjoint from the minimal section (the node) and that section. If S denotes a section with $S^2 = 2$, and F a fibre, the branch curve is simply $5S + (S - 2F)$. As $K = -2B$ we have $K + B = S - F = (S - 2F) + F$ with $S - 2F$ as fixed component. Thus the free part of the canonical pencil gives a genus two fibration.

A generic quadric in the pencil intersects the sextic in a $(6,6)$ curve which splits up into the fixed component (the $(2,2)$ curve) and a residual $(4,4)$ curve with seven double points, passing simply through the node P_8 of the $(2,2)$ curve.

The genus of the desingularisation is indeed $3 \times 3 - 7 = 2$, giving us the genus two pencil.

In fact if $\{q + tq_0 = 0\}$ is a quadric in the pencil, the residual curve is the intersection $\{q + tq_0 = tq^2 + g = 0\}$. The canonical system on such a genus two curve is given by adjunction as the pencil of $(2, 2)$ curves passing through the seven nodes. Thus the involution on the surface will be given as follows. For each point P choose t such $q + tq_0$ vanish at P, then consider the residual intersection with the base locus of the pencil of quadrics through $P_1, \dots, P_7, P$ and the quartic $\{tq^2 + g = 0\}$.

One (-1)**-conic:** such a sextic can be obtained as the reciprocal transform of the previous sextic in P_1, P_2, P_3 and P_8 (which necessarily do not lie in one plane). The quadric Q transforms into a smooth quadric $Q' = \{q' = 0\}$, Q_0 transforms into a plane $H'_0 = \{h'_0 = 0\}$ and Y transforms into a quintic $Y' = \{g' = 0\}$ with three triple points and five double points. So the transform X' of X satisfies the equation $h'_0 g' + {q'}^3 = 0$. The image of C is a (-1)-conic lying in the plane H'_0. Conversely, every such sextic can be transformed into one with a (-1)-curve of type $(2, 2)$: just take as fundamental points the three triple points not on H'_0 and a fourth triple point on H'_0. The base locus of the pencil of quadrics consists of the (-1)-conic and another conic not contained in X'. This family again has 13 moduli.

Two (-1)**-conics:** the sextics X with two (-1)-conics C_1 and C_2 are easily identified as those satisfying an equation of the form $h_1h_2g + q^3 = 0$. Here $H_i = \{h_i = 0\}$ are planes containing C_i, $i = 1, 2$. By corollary 4.8, the two conics intersect in two triple points, say P_1 and P_2. Then $Y = \{g = 0\}$ is a quartic surface through P_1 and P_2 with only nodes in $P_3, \dots, P_8$ and $Q = \{q = 0\}$ is a general element of the pencil. There are ten linearly independent sextics with the eight triple points: four from the pencil spanned by h_1h_2 and q, another four of the form $h_1^2h_2^2q_1$ where q_1 is one of the four quadrics through $P_3, \dots, P_8$ and finally $h_1h_2^2k_1$, $h_1^2h_2k_2$ for cubics k_1 and k_2 through all eight points such that k_i has double points in the triple points not contained in h_i, $i = 1, 2$. The point configuration has five moduli, so we get a 14 parameter family. Again every such sextic is of general type.

The minimal models will have $p_g = 2$ and $c_1^2 = 2$ and come equipped with genus three fibrations. Those will be defined by, in analogy with the case of a (-1) curve of type $(2, 2)$, as residual $(4, 4)$ curves, with nodes at $P_3 \dots P_8$, passing through P_1, P_2. We leave it to the reader to work out the details.

No (-1)**-curves:** it is not so immediate to construct such surfaces. We follow the classical construction for asyzygetic eight-nodal quartics [C, R], which we now recall. Seven points $P_1, \dots, P_7$ in general position define a net of quadrics spanned by smooth quadrics $Q_i = \{q_i = 0\}$, $i = 1, 2, 3$. All quartics with nodes in $P_1, \dots, P_7$ are given by $g + f_2(q_1, q_2, q_3) = 0$, where f_2 is a ternary quadratic form and g is a fixed nodal quartic, which we can take as the product of a cubic with

nodes in $P_1, \dots, P_4$ passing simply through P_5, P_6, P_7 and a plane through P_5, P_6, P_7. A sufficient condition for an eighth singular point is that the derivatives dg, dq_1, dq_2 and dq_3 are linearly dependent. The vanishing of the determinant

$$\begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial q_1}{\partial x} & \frac{\partial q_2}{\partial x} & \frac{\partial q_3}{\partial x} \\ \frac{\partial g}{\partial y} & \frac{\partial q_1}{\partial y} & \frac{\partial q_2}{\partial y} & \frac{\partial q_3}{\partial y} \\ \frac{\partial g}{\partial z} & \frac{\partial q_1}{\partial z} & \frac{\partial q_2}{\partial z} & \frac{\partial q_3}{\partial z} \\ \frac{\partial g}{\partial w} & \frac{\partial q_1}{\partial w} & \frac{\partial q_2}{\partial w} & \frac{\partial q_3}{\partial w} \end{vmatrix}$$

defines a degree six surface Δ, called the Cayley dianode surface. It is the closure of the locus of points P_8 such that there exists a quartic surface with only nodes in $P_1, \dots, P_8$. In general the dianode surface has triple points in $P_1, \dots, P_7$ as its only singularities, but it can reducible: if four points lie in a plane this plane becomes a component.

We now look at sextics with seven triple points in general position. We find 14 such sextic equations: the ten third powers of the three quadrics q_1, q_2 and q_3, three equations of the form $\{gq_i = 0\}$ for a quartic g which is *not* a conic of the three quadrics and finally a sextic δ not of this form, for which one can take the Cayley dianode surface. The vanishing of all second derivatives in an eighth point P_8 gives ten linear equations in the 14 coefficients. Note however that the columns of the coefficient matrix are not linearly independent. We observe that the all second derivatives of a third power have the function itself as factor:

$$(f^3)_{ij} = 3f(2f_i f_j + f f_{ij}) \; .$$

For a product we have

$$(f^2 g)_{ij} = (2f_i f_j + f f_{ij})g + f(2f_i g_j + 2f_j g_i + f_{ij} g + f g_{ij})$$

and a similar expression for the derivatives of fgh. Suppose that $f \neq 0$. Then we can divide all partials $(f^3)_{ij}$ by f and after subtracting the same multiple of $2f_i f_j + f f_{ij}$ from all partials $(f^2 g)_{ij}$ we can again divide by f. From $(fg^2)_{ij}$ we can get $2g_i g_j + g g_{ij}$. So if one of the quadrics does not vanish (i.e., the point is not the 8th base point of the net), the 10 columns of the third powers give at most 6 independent ones and our matrix reduces to a 10×10 matrix. One obtains a determinant of degree 28, but the rank of the matrix is eight on the dianode surface, whose equation is double factor of the determinant. We are left with a surface Δ' of degree 16. It has as double curves the 21 lines joining the triple points, the 7 cubics though 6 of the 7 points and the Steiner curve of the net. The dianode surface Δ intersects Δ' exactly in its singular locus.

Now we get two possibilities to construct a sextic, by taking P_8 in Δ or in Δ'. If P_8 is a general point of the Cayley dianode surface, there exists a quartic surface $Y = \{g = 0\}$ with only nodes in $P_1, \dots, P_8$. Then a general linear combination of gq_1, gq_2, q_1^3, $q_1^2 q_2$, $q_1 q_2^2$, q_2^3 defines a sextic X with only triple points in the eight

points and containing the base locus of the pencil spanned by Q_1 and Q_2. The surface will be a (minimal) elliptic surface, whose elliptic pencil is given by the pencil of quadrics through the points $P_1, \ldots, P_8$: the residual intersection on a smooth quadric of the pencil is a curve of type $(4, 4)$ with 8 double points. The point configuration having eight moduli, we get a 13 parameter family.

Choosing $P_8 \in \Delta'$ general we obtain a sextic with eight triple points not containing parts of the base locus of the pencil. The number of parameters is now $8 + 4 = 12$. The surface is again a (minimal) elliptic surface, with elliptic pencil given by the pencil of quadrics: a general intersection is a $(6, 6)$-curve with eight triple points, hence the geometric genus is one.

4.4 Nine triple points

Assume that X has $\nu = 9$ triple points $P_1, \ldots, P_9$. Let Q be the unique (irreducible) canonical quadric surface and let $K = Q \cdot X$ be the adjoint curve. By propositions 4.10 and 4.9 the resolution $\tilde{X}$ has exactly three disjoint (-1)-curves C_1, C_2, C_3. If $C_1 + C_2 + C_3 = K$, blowing down C_1, C_2 and C_3 gives a minimal surface $\overline{X}$ with $c_1\left(\overline{X}\right)^2 = 0$, $c_2\left(\overline{X}\right) = 24$ and $K_{\overline{X}} = \mathcal{O}_{\overline{X}}$. Then $\tilde{X}$ is a $K3$ surface blown up in three points. Otherwise we end up with an effective canonical divisor after blowing down C_1, C_2 and C_3. This implies $\kappa(\tilde{X}) = 1$ and thus $\tilde{X}$ is the blowup of a minimal properly elliptic surface in three points. As it will have the same basic invariants as a $K3$ surface, it will be obtained from an elliptic such by a series of logarithmic transforms, i.e. making some elliptic fibres multiple.

The sextic $K3$ surface

Using corollary 4.8, the multiplicities of the (-1)-curves in the nine triple points are easily found to be (up to a permutation of triple points) as in the following table.

type		P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
$(4,4,4)$	C_1	2	1	0	1	1	1	1	1	1
	C_2	0	2	1	1	1	1	1	1	1
	C_3	1	0	2	1	1	1	1	1	1
$(2,4,6)$	C_1	1	0	0	0	0	1	1	1	1
	C_2	0	2	1	1	1	1	1	1	1
	C_3	2	1	2	2	2	1	1	1	1
$(2,2,8)$	C_1	1	0	0	0	0	1	1	1	1
	C_2	0	1	1	1	0	0	0	1	1
	C_3	2	2	2	2	3	2	2	1	1

Table 5: multiplicities of the (-1)-curves

Proposition 4.12 *If* $(c_1, c_2, c_3) = (4, 4, 4)$*, then* X *satisfies an equation of the form*

$$q_1 q_2 q_3 + q^3 = 0 \ ,$$

where $Q = \{q = 0\}$ *is the unique canonical surface. Each* $Q_i = \{q_i = 0\}$ *is a quadric cone through eight triple points with vertex in one of them and* $C_i = Q \cdot Q_i$*,* $i = 1,\ 2,\ 3$*.*

Proof: The three quartic (-1)-curves are of type $(2, 2)$ by lemma 4.7. Lemma 4.11 guarantees the existence of three quadric cones $Q_i = \{q_i = 0\}$ such that $X \cdot Q_i = 3C_i$, $i = 1, 2, 3$. Thus $X \cdot (Q_1 + Q_2 + Q_3) = 3(C_1 + C_2 + C_3)$, hence a equation of X and $q_1q_2q_3$ differ by the cube of a quadratic polynomial vanishing in all triple points. Since $p_g(\tilde{X}) = 1$, such polynomial defines the unique canonical surface. □

Let $Q_i = \{q_i = 0\}$ be the quadric cones with vertices P_i such that Q_i passes through $P_{i+1 \bmod 3}$ but not through $P_{i+2 \bmod 3}$, $i = 1, 2, 3$. We choose P_1, P_2 and P_3 as $(1:0:0:0)$, $(0:1:0:0)$ and $(0:0:1:0)$. If we require that the quadrics also pass through $(0:0:0:1)$ and $(1:1:1:1)$ their equations can be written (inhomogeneously) as

$$\begin{aligned} q_1 &= z^2 + a_1 y + b_1 z - (a_1 + b_1 + 1)yz, \\ q_2 &= x^2 + a_2 z + b_2 x - (a_2 + b_2 + 1)xz, \\ q_3 &= y^2 + a_3 x + b_3 y - (a_3 + b_3 + 1)yx, \end{aligned}$$

where the a_i and b_i are constants. The quadrics intersect in a zero-dimensional scheme of length 8 containing at least six distinct points. To find the canonical surface $Q = \{q = 0\}$ we have to pick six points $P_4, \ldots\ P_9$. It is easier to specify the remaining scheme of length two. For a Zariski open set of the stratum it will consist of two points in general position, which we take as $(0:0:0:1)$ and $(1:1:1:1)$. Thus we require that Q does not pass through these two points. To compute its equation $q = 0$ we note that the q_i lie in the ideal defining $(1:1:1:1)$; they can be written in vector form as

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 & (b_1 + z)z & (a_1 - z)y \\ (a_2 - x)z & 0 & (b_2 + x)x \\ (b_3 + y)y & (a_3 - y)x & 0 \end{pmatrix} \begin{pmatrix} 1 - x \\ 1 - y \\ 1 - z \end{pmatrix} .$$

In the triple points of X the q_i vanish, while $(x, y, z) \neq (1, 1, 1)$. So the determinant of the matrix vanishes. Dividing it by xyz gives the inhomogeneous equation

$$q = (a_1 - z)(a_2 - x)(a_3 - y) + (b_1 + z)(b_2 + x)(b_3 + y)$$

which is indeed the sought after quadric (note that the two terms xyz cancel). The general element of the pencil $\alpha q_1 q_2 q_3 + \beta q^3$ defines a sextic X with nine triple points with multiplicities as in the first part of table 5.

A particular example is obtained by taking $b_i = 0$, $a_i = -1$, so $(q_1, q_2, q_3) = (z^2 - y, x^2 - z, y^2 - x)$ and $P_{3+i} = (\eta^{4i}, \eta^{2i}, \eta^i)$ with η a primitive seventh root of unity. The number of parameters in the construction is 7 (the a_i, b_i and $(\alpha : \beta)$).

Now consider the reciprocal transformation with fundamental points P_1, P_2, P_4 and P_5. In general the image X' of X will have nine ordinary triple points as only singularities. It can be checked that this is indeed the case for the particular example, where we take (η^4, η^2, η) and $(\overline{\eta}^4, \overline{\eta}^2, \overline{\eta})$ as third and fourth fundamental point. Then the image H_1' of Q_1 is a plane, the image Q_2' of Q_2 is a quadric cone and the image Y_3' of Q_3 is a cubic surface with four nodes. The image Q' of Q being again a quadric, the sextic X' is given by an equation of the form

$$h_1' q_2' g_3' + {q'}^3 = 0.$$

Note that C_1, C_2 and C_3 are mapped onto (-1)-curves C_1', C_2' and C_3' of degree 2, 4 and 6.

Surfaces with an equation of this type form a seven dimensional family. The nodes of the cubic, the vertex of the quadric and two points in the plane can be specified, while the last two are then to be found among the four other intersection points of the cubic, the cone and the plane. The dimension of the Zariski tangent space to the equisingular stratum in the specific example is $15 + 7$, so we obtain a full seven parameter family of sextics with nine triple points and $(c_1, c_2, c_3) = (2, 4, 6)$.

Now let X have $(c_1, c_2, c_3) = (2, 4, 6)$ and an equation of the form $h_1 q_2 g + q^3 = 0$, where h_1 determines a plane H_1, q_2 a quadric cone Q_2 and g_3 a four-nodal cubic Y_3. Consider the reciprocal transform with fundamental points P_2, P_5, P_6 and P_7 as in table 5. Continuing with the specific example above, one can take the images of the points (η, η^4, η^2) and $(\overline{\eta}, \overline{\eta}^4, \overline{\eta}^2)$ as P_6 and P_7. The reciprocal image has nine ordinary triple points, so again this property holds on an open dense set in the parameter space of all sextics with $(c_1, c_2, c_3) = (2, 4, 6)$. Here the image H_1' of H_1 is again a plane, the image H_2' of Q_2 is also a plane and the image Y_4' of Y_3 is a quartic surface with one triple point and six double points. The image Q' of Q is again a quadric, so the image X' of X is given by an equation of the form

$$h_1' h_2' g_4' + {q'}^3 = 0.$$

This time the curves C_1, C_2 and C_3 are mapped onto (-1)-curves C_1', C_2' and C_3' of degrees 2, 2 and 8.

Once more we check that this is a full seven parameter family of sextic surfaces with nine triple points and $(c_1, c_2, c_3) = (2, 2, 8)$. A direct construction of the family starts with seven points in general position, of which we choose one as

the triple point for the quartic Y_4 and divide the remaining six into two groups of three, each determining a plane H_i. The intersection $Y_4 \cap H_1 \cap H_2$ consists of four points. Two of them can be triple points for a sextic (remember that three are not allowed on a line) in the pencil $\alpha q^3 + \beta h_1 h_2 g_4$. An explicit example starts coordinate vertices and the points $(\lambda : 1 : 1 : 1)$, $(1 : \lambda : 1 : 1)$ and $(1 : 1 : \lambda : 1)$. We get

$$\begin{aligned} h_1 = t, \qquad h_2 &= x + y + z - (\lambda + 2)t \\ g_4 = \lambda(\lambda+1)(\lambda+2)(xy+xz+yz)^2 &- (2\lambda+1)^2(\lambda+1)xyz(x+y+z) \\ -\lambda(2\lambda+1)(xy+xz+yz)(x+y+z)t &+ (2\lambda+1)^2(\lambda+2)xyzt\ . \end{aligned}$$

The intersection line $H_1 \cap H_2$ is now a double tangent of $Y_4 \cap H_1$ so we take the two points of tangency as last two triple points. We find

$$q = (2+\lambda)(xy+xz+yz) - (2\lambda+1)(\lambda+1)(x+y+z)t\ .$$

There is a certain amount of choice in the fundamental points of the reciprocal transformations. They depend also on the position of the triple points. So we can conclude that we obtain correspondences between our families of sextics with triple points, whose exact nature we did not determine. We will say that our families are 'related via reciprocal transformations'.

We summarise our findings.

Theorem 4.13 *For every $(c_1, c_2, c_3) \in \{(2,2,8), (2,4,6), (4,4,4)\}$ there exists a seven parameter family of sextic surfaces with nine triple points and three (-1)-curves of degrees c_1, c_2 and c_3. The three families are related via reciprocal transformations. For every such surface X its minimal desingularisation $\tilde{X}$ is a $K3$ surface blown up in three points. Moreover X satisfies an equation of the form*

$$\begin{aligned} q_1q_2q_3 + q^3 = 0 &\quad \text{if } (c_1, c_2, c_3) = (4,4,4). \\ h_1q_2g_3 + q^3 = 0 &\quad \text{if } (c_1, c_2, c_3) = (2,4,6), \\ h_1h_2g_4 + q^3 = 0 &\quad \text{if } (c_1, c_2, c_3) = (2,2,8). \end{aligned}$$

Here $Q = \{q = 0\}$ is the unique canonical surface. The three exceptional curves are in each case obtained as the locus where one of the three forms in the product and q vanish. In the last case g_4 defines a quartic surface with a triple point six double points. In the second case, g_3 defines a four nodal cubic.

It may now be amusing to digress on the geometry of the $K3$ surfaces obtained. Consider the case $(4,4,4)$. Let E_i be the image in the minimal $K3$ surface of the exceptional curve in the resolution of P_i. Notice that $E_i^2 = 2$ if $i \le 3$ and $E_i^2 = 0$ otherwise. Furthermore for $i \ne j$ we have $E_i \cdot E_j = 2$ if $i, j \le 3$ while $E_i \cdot E_j = 3$ otherwise. For future reference let us denote the case $E^2 = 0$ as the

first type and $E^2 = 2$ as the second type, and by slight abuse of terminology, also the corresponding triple points.

It is now easy to write down

$$2H = \sum_i E_i - 8(C_1 + C_2 + C_3)$$

and from the above it is straightforward to check that $(2H)^2 = 24$. Notice also that $\sum_i E_i$ is an even divisor.

Conversely given such a configuration of curves E_i in the Picard group we need to choose the points c_i to be blown up carefully. Any divisor E with $E^2 = 2$ on a $K3$ surface determines a net, and thus a Jacobian curve of the net, corresponding to the curve-locus of singular points of singular members. Each E_i, $i \leq 3$ determines such a curve J_i. The point c_1 has to be chosen on J_1. The point c_2 has to lie on the intersection of J_2 with some element in $|E_1|$ singular at c_1 and we expect only a finite number of such choices. Furthermore c_3 has to lie on the intersection of J_3 with some element of $|E_2|$ singular at c_2, and finally some element of $|E_3|$ singular at c_3 should pass through c_1. This indicates that c_1 should be chosen with care, and you thus expect only a finite number of configurations of points c_i. Further conditions are that the remaining E_i also pass through the points c_i. We expect a 11-dimensional family of $K3$ surfaces with the appropriate sublattices, and therefore four independent conditions. Their exact nature remains mysterious.

To consider the two remaining cases, one writes down respectively

$$2H = \sum_i E_i - 4C_1 - 8C_2 - 12C_3$$

and

$$2H = \sum_i E_i - 4C_1 - 4C_2 - 16C_3$$

with different intersection matrices, and similar conditions on the points c_i; in the last case there will also be a third type of elliptic curve ($E_5^2 = 6$).

To see how those are related to the first case we note (once again) the fact that the intersection of a sextic with a plane passing through exactly three triple point gives an elliptic curve F with $F^2 = -3$ after the resolution. Given a tetrahedron of triple points, thus means replacing the exceptional vertices with the elliptic curves of the faces.

To make this explicit return to the first case $(4, 4, 4)$. Let us denote by i, j, k different integers strictly greater than three, and m an integer less or equal to three. We get four cases for F namely

$$F = H + 3(C_1 + C_2 + C_3) - E_i - E_j - E_k$$

or

$$F = H + 3(C_1 + C_2 + C_3) - E_1 - E_2 - E_3$$

with $F \cdot C_m = 1$ and also

$$F = H + 4C_1 + 2C_2 + 3C_3 - E_1 - E_j - E_k$$

or

$$F = H + 4C_1 + 3C_2 + 2C_3) - E_1 - E_2 - E_k$$

with $F \cdot C_1 = 0$, $F \cdot C_2 = 2$, $F \cdot C_3 = 1$ and $F \cdot C_1 = 0$, $F \cdot C_2 = 1$, $F \cdot C_3 = 2$ respectively.

Thus in the first two cases we get elliptic curves F of the first type, while in the last two cases, curves of the second type.

In the case $(4, 4, 4)$ we have three curves of the second type and six of the first. If we choose a tetrahedron with no triple points of the second type, or three, the number of elliptic curves of first or second type will not change. However if there are one or two triple points of the second type, after the transformation the number of each type will change from $6, 3$ to $4, 5$ landing us in the case $(2, 4, 6)$. We leave it to the reader to continue the analysis and show how we can get from situation $(2, 4, 6)$ back to $(4, 4, 4)$ or to $(2, 2, 8)$.

The sextic properly elliptic surface

Now we turn to the remaining cases where $C_1 + C_2 + C_3$ does not make up K. Using corollary 4.8 again, the multiplicities of the (-1)-curves in the nine triple points are easily found to be (up to a permutation of triple points) as in the following table.

type		P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
	C_1	0	0	1	1	1	1	1	0	0
$(2, 2, 2)$	C_2	1	1	0	0	1	1	0	1	0
	C_3	1	1	1	1	0	0	0	0	1
	C_1	0	0	1	1	1	1	1	0	0
$(2, 2, 4)$	C_2	1	1	0	0	1	1	0	1	0
	C_3	1	1	1	1	0	1	1	1	2

Table 6: multiplicities of the (-1)-curves

Let us construct a sextic X with nine triple points and $(c_1, c_2, c_3) = (2, 2, 2)$. Then X has an equation of the form $f = h_1 h_2 h_3 g + q^3 = 0$. So take three planes in general position $H_i = \{h_i = 0\}$, $i = 1, 2, 3$ and let $L_1 = H_1 \cap H_2$, $L_2 = H_1 \cap H_3$ and $L_3 = H_2 \cap H_3$ be the lines of intersection. On each line L_i choose two different points P_{2i-1}, P_{2i}. Finally choose points $P_{i+7} \in H_i$ not on the lines. The nine points determine a unique quadric $Q = \{q = 0\}$. Let $H_4 = \{h_4 = 0\}$ be the plane through P_7, P_8 and P_9. The reducible cubic $qh_4 = 0$ is an element of the pencil of

cubics through all points with double points in P_7, P_8 and P_9. Let $Y = \{g = 0\}$ be another such cubic. Then the general element of the net

$$\alpha q^3 + \beta h_1 h_2 h_3 q h_4 + \gamma h_1 h_2 h_3 g$$

has only isolated triple points at the nine points.

Note that the canonical divisor will be a $(3,3)$ curve with three nodes, the intersection of Y with Q. This is an elliptic curve and constitutes a reduced multiple fibre F_0. As $K = (m-1)F_0$ we conclude $m = 2$. (In fact we have, as X has the same invariants as a $K3$ surface, $K = \sum_i (m_i - 1)F_{m_i}$, where all the multiple fibres F_{m_i} are fixed components of the canonical divisor) Thus we have blown up a so called 'fake $K3$' surface, obtained from an elliptic $K3$ surface, by making a fibre double. Thus in particular $P_2 = 2$ as $2K = F$. In fact the bicanonical divisors are given by quartics with nodes at the nine triple points. We can write down two independent such quartics, namely q^2 and $h_1 h_2 h_3 h_4$. The pencil spanned by those cuts out the residual to the configuration of the three exceptional conics (with multiplicity two). As we have noted q^2 cuts out the double fibre F_0, while the other quartic will cut out the elliptic curve F (intersection with the plane h_4) along with the three exceptional divisors. Thus F is a general fibre, which has been blown up three times.

It could be interesting to investigate the position of the resolution curves E_i (corresponding to the triple points P_i). Three of those E_7, E_8, E_9 span the plane h_4 and thus each of them intersects F three times. They also intersect F_0 twice (corresponding to its nodes). Furthermore each of them intersects exactly one of the exceptional divisors, namely the one, whose plane the corresponding triple point happens to lie on. Thus in the minimal model these curves are elliptic four-sections, thus self-intersection -2. The remaining six intersect F_0 simply, hence are bi-sections, meaning self-intersection -1. Each of those intersects exactly two exceptional divisors and in the minimal model they will intersect F twice rather than being disjoint from it, as in the resolution.

The reader could easily work out the intersection matrix of those nine curves and from that write down the hyperplane section. A similar analysis could be made for the second case (to be discussed below), and the relation between the two via the reciprocal transformation, elucidated by considering the possible tetrahedrons of triple points.

In order to search for a tenth triple point we need explicit equations. After a change of coordinates we may assume that the planes are the sides of the coordinate tetrahedron. The remaining coordinate transformations are given by diagonal matrices. We take $P_7 = (0:1:\lambda:0)$, $P_8 = (\mu:0:1:0)$ and $P_9 = (1:\nu:0:0)$. A cubic through these points and also through $(0:0:0:1)$ is

$$\begin{aligned} g = {} & w^2(a_1 x + a_2 y + a_3 z) \\ & + w(a_4 x(\nu x - y - \mu\nu z) + a_5 y(\lambda y - \lambda\nu x - z) + a_6 z(\mu z - x - \mu\lambda z)) \\ & + (\nu x - y - \mu\nu z)(\lambda y - \lambda\nu x - z)(\mu z - x - \mu\lambda z), \end{aligned}$$

while a quadric through P_7, P_8 and P_9 is

$$\begin{aligned} q =& b_4b_5b_6w^2 + w(b_1b_4x + b_2b_5y + b_3b_6z) \\ &+ b_4x(\nu x - y - \mu\nu z) + b_5y(\lambda y - \lambda\nu x - z) + b_6z(\mu z - x - \mu\lambda z), \end{aligned}$$

where the parameters are chosen with hindsight. The condition that Y and Q intersect the coordinate axes in the same points leads to easy equations between the coefficients. We get

$$\begin{aligned} g =& w^2(\lambda\nu b_5b_6x + \lambda\mu b_4b_6y + \mu\nu b_4b_5z) \\ &+ w\big(\lambda b_1x(\nu x - y - \mu\nu z) + \mu b_2y(\lambda y - \lambda\nu x - z) + \nu b_3z(\mu z - x - \mu\lambda z)\big) \\ &+ (\nu x - y - \mu\nu z)(\lambda y - \lambda\nu x - z)(\mu z - x - \ mu\lambda z) \ . \end{aligned}$$

The formulas contain nine parameters, three of which can be removed by coordinate transformations. The family depends therefore on eight moduli.

We can specialise to a symmetric equation by taking $\lambda = \mu = \nu = 1$, $b_1 = b_2 = b_3 = b$ and $b_4 = b_5 = b_6 = \sqrt{a}$. With $a = b = 1$ the surface $X = \{xyzg + q^3 = 0\}$ is a sextic with only nine triple points.

In this example we can compute the tangent space to the equisingular stratum to have dimension $15 + 8$. This shows that our construction fills up a whole component. Note that the multiplicities of the (-1)-curves are just as in table 6.

Consider the reciprocal transformation centred in P_6, P_7, P_8 and P_9. The image H_1' of H_1 is a plane, the image H_2' of H_2 is a plane, the image Q_3' of H_3 is a quadric cone and the image Q_4' of K is a smooth quadric. So the image X' of X is given by an equation

$$\alpha q'^3 + \beta h_1'h_2'q_1'q' + \gamma h_1'h_2'q_1'q_2' = 0 \ .$$

The images C_1', C_2' and C_3' of C_1, C_2 and C_3 are (-1)-curves of degrees 2, 2 and 4. As before we see that the two cases $(c_1, c_2, c_3) = (2, 2, 2)$ and $(c_1, c_2, c_3) = (2, 2, 4)$ are related by reciprocal transformations. So there exists an eight parameter family of sextics with nine triple points and $(c_1, c_2, c_3) = (2, 2, 4)$.

Now the canonical divisor will be a smooth $(2, 2)$ curve. It is again a fibre occurring with multiplicity two.

We summarise:

Theorem 4.14 *For every $(c_1, c_2, c_3) \in \{(2, 2, 2), (2, 2, 4)\}$ there exists an eight parameter family of sextic surfaces with nine triple points and three (-1)-curves of degrees c_1, c_2 and c_3. The two families are related via reciprocal transformations. For every such surface X its minimal desingularisation $\tilde{X}$ is a minimal properly elliptic surface blown up in three points. Moreover X satisfies an equation of the form*

$$h_1h_2h_3g + q^3 = 0 \quad \textit{if } (c_1, c_2, c_3) = (2, 2, 2),$$
$$h_1h_2q_3g + q^3 = 0 \quad \textit{if } (c_1, c_2, c_3) = (2, 2, 4).$$

Here $Q = \{q = 0\}$ is the unique canonical surface. The three exceptional curves are in each case obtained as the locus where the three forms in the product and q vanish. In the first case g defines a cubic with three double points. In the second case, g defines a smooth quadric.

4.5 Ten triple points

Let X be a sextic with $\nu = 10$ triple points. As for the type of $\tilde{X}$ in the classification, we have the

Proposition 4.15 *If $\nu = 10$, then X is rational.*

Proof: We will show that $\kappa(\tilde{X}) = -\infty$, then the result follows from the Enriques-Kodaira classification.

Assume that $\kappa(\tilde{X}) = 0$. Then $\tilde{X}$ would be an Enriques surface $\overline{X}$ blown up in six points. Hence $P_2(\tilde{X}) = 1$, so there exists a unique quartic bicanonical surface Y intersecting X in a degree 24 curve D made up by the six (-1)-curves $C_1, \ldots, C_6$ of degrees $c_1, \ldots, c_6$ (remember $2K_{\overline{X}} = \mathcal{O}_{\overline{X}}$). On the one hand we get $\mathrm{mult}(D, \mathcal{S}) = 2(c_1 + \ldots + c_6) + 6 = 54$ from proposition 3.3. On the other hand $2K_{\tilde{X}} \sim_{lin} 4H - 2E$, hence $\mathrm{mult}(D, \mathcal{S}) = 10 \cdot 2 \cdot 3 = 60$, contradiction.

Now assume that $\kappa(\tilde{X}) \geq 1$, then $P_2(\tilde{X}) \geq 2$. So we have at least two quartic bicanonical surfaces Y_1 and Y_2 intersecting in a degree 16 curve D. We must have $\mathrm{mult}(D, \mathcal{S}) = 10 \cdot 2 \cdot 2 = 40$. There exists a decomposition $D = D_1 + D_2$ with $D_1 \subset X$ and no component of $D_2 \neq 0$ is contained in X. Let $d_i = \deg D_1$, $i = 1, 2$. Then $\mathrm{mult}(D, \mathcal{S}) = \mathrm{mult}(D_1, \mathcal{S}) + \mathrm{mult}(D_2, \mathcal{S}) \leq 5d_1/2 + 2d_2 < 40$, contradiction. $\square$

Every sextic with ten triple points is a specialisation of a family of sextics with nine triple points: if $Q = \{q = 0\}$ is the unique quadric through nine out of the ten triple points, the general element of the pencil $\alpha f + \beta q^3 = 0$ is a sextic with nine triple points, where f is a defining equation for X. A sextic with $\nu = 10$ is likely to be found in any of the five families with nine triple points described above, as a triple point gives seven conditions. However the equations on the coefficients become rather formidable, and we have only succeeded in one case by imposing extra symmetry. We start with the first family of properly elliptic surfaces with equations

$$\alpha q^3 + \beta xyzqw + \gamma xyzg$$

where the planes $H_1 = \{x = 0\}, \ldots, H_4 = \{w = 0\}$ are the faces of the coordinate tetrahedron and the cubic and quadric are as given above. We use the remaining freedom in coordinate transformations to place the putative tenth triple point in $(1:1:1:1)$. We compute in the affine chart $w = 1$.

The condition for a triple point is then that the function, its derivatives and the second order derivatives vanish at $(1, 1, 1)$. This gives ten equations which

are linear in α, β and γ, so we may eliminate them: the maximal minors of the coefficient matrix have to vanish. We have

$$\frac{\partial\, xyzg}{\partial x} = yzg + xyzg_x, \qquad \frac{\partial^2\, xyzg}{\partial x^2} = 2yzg_x + xyzg_{xx} \quad \text{and}$$
$$\frac{\partial^2\, xyzg}{\partial x\, \partial y} = zg + xzg_x + yzg_y + xyzg_{xy}.$$

Now we plug in $x = y = z = 1$. From g we get

$$\begin{aligned}
&\lambda\nu b_5 b_6 + \lambda\mu b_4 b_6 + \mu\nu b_4 b_5 \\
&+ \lambda b_1(\nu - 1 - \mu\nu) + \mu b_2(\lambda - \lambda\nu - 1) + \nu b_3(\mu - 1 - \mu\lambda) \\
&+ (\nu - 1 - \mu\nu)(\lambda - \lambda\nu - 1)(\mu - 1 - \mu\lambda),
\end{aligned}$$

an expression which we continue to denote by g. We get also expressions for all derivatives. Likewise we have

$$\begin{aligned}
q =\,& b_4 b_5 b_6 + (b_1 b_4 + b_2 b_5 + b_3 b_6) \\
&+ b_4(\nu - 1 - \mu\nu) + b_5(\lambda - \lambda\nu - 1) + b_6(\mu - 1 - \mu\lambda)\ .
\end{aligned}$$

Moreover

$$\frac{\partial q^3}{\partial x} = 3q^2 q_x, \qquad \frac{\partial^2 q^3}{\partial x^2} = 3q^2 q_{xx} + 6qq_x^2 \quad \text{and}$$
$$\frac{\partial^2 q^3}{\partial x \partial y} = 3q^2 q_{xy} + 6qq_x q_y.$$

All these are divisible by q. After dividing by q our matrix has the following form:

$$\begin{pmatrix}
q^2 & 3qq_x & \dots & 3qq_{xx} + 6q_x^2 & \dots & 3qq_{xy} + 6q_x q_y & \dots \\
q & q + q_x & \dots & 2q_x + q_{xx} & \dots & q + q_x + q_y + q_{xy} & \dots \\
g & g + g_x & \dots & 2g_x + g_{xx} & \dots & g + g_x + g_y + g_{xy} & \dots
\end{pmatrix}$$

We simplify this matrix by subtracting $3q$ times the second row from the first row to remove all second derivatives from the first row. After that we apply only column operations. A computation reveals that $q_{xx} + 2\nu q_{xy} + \nu^2 q_{yy} = 0$ and also $g_{xx} + 2\nu g_{xy} + \nu^2 g_{yy} = 0$. Analogous equations hold for the other second partials. After multiplying the column containing q_{xy} by ν and further column operations we get

$$\begin{pmatrix}
-2q^2 & -q^2 & \dots & 2q^2 - 6qq_x & \dots & p_{xy} & \dots \\
q & q_x & \dots & q_{xx} & \dots & 0 & \dots \\
g & g_x & \dots & g_{xx} & \dots & 0 & \dots
\end{pmatrix},$$

where

$$p_{xy} = (\nu^2+\nu+1)q^2 - 3(\nu+1)q(\nu q_y + q_x) + 3(\nu q_y + q_x)^2 \,.$$

Now the entries in the columns with two zeroes have to vanish, for otherwise $\alpha = 0$ and the equation for the sextic is divisible by xyz. We obtain the three equations

$$\begin{aligned}
(\nu^2+\nu+1)q^2 - 3(\nu+1)q(\nu q_y + q_x) + 3(\nu q_y + q_x)^2 &= 0\,,\\
(\mu^2+\mu+1)q^2 - 3(\mu+1)q(\mu q_x + q_z) + 3(\mu q_x + q_z)^2 &= 0\,,\\
(\lambda^2+\lambda+1)q^2 - 3(\lambda+1)q(\lambda q_z + q_y) + 3(\lambda q_z + q_y)^2 &= 0\,.
\end{aligned}$$

This shows that the locus we are after consists of several components. The discriminant of the first equation, as a quadratic form in q and $\nu q_y + q_x$, equals $-3(\nu-1)^2$, which implies that no solution is defined over $\mathbb{R}$. In principle q_x is now expressible in terms of λ, μ, ν and q, so the first row becomes divisible by q^2. We may divide by q^2 because there is no quadric through all ten triple points.

This simplification is not enough to solve the equations. To obtain manageable equations we impose symmetry. We take $\lambda = \mu = \nu$, $b_1 = b_2 = b_3 = b$ and $b_4 = b_5 = b_6 = \sqrt{a}$. This gives

$$\begin{aligned}
g =\, &\lambda^2 a(x+y+z) + b\left(x(\lambda x - y - \lambda^2 z) + y(\lambda y - \lambda^2 x - z) + z(\lambda z - x - \lambda^2 y)\right)\\
&+ (\lambda x - y - \lambda^2 z)(\lambda y - \lambda^2 x - z)(\lambda z - x - \lambda^2 y)
\end{aligned}$$

and after dividing by $\sqrt{a}$

$$q = a + b(x+y+z) + x(\lambda x - y - \lambda^2 z) + y(\lambda y - \lambda^2 x - z) + z(\lambda z - x - \lambda^2 y).$$

The second derivatives evaluated in $(1,1,1)$ give only four different equations due to the symmetry in x, y and z. Our three equations reduce to

$$\frac{1}{4}(\lambda-1)^2 q^2 + 3(\lambda+1)^2(q_x - \frac{1}{2}q)^2 = 0\,.$$

Substituting the values of q and q_x gives up to a constant

$$3(\lambda-1)^2(b + a/3 - \lambda^2 + \lambda - 1)^2 + (\lambda+1)^2(b + a + \lambda^2 - \lambda + 1)^2 = 0\,.$$

The first seven columns of our matrix above reduce to three independent ones. After multiplication of the last column by $(\lambda+1)^2$ we can use the equation above to eliminate q_x. Upon division by λ we get

$$\begin{pmatrix} -2q^2 & -q^2 & -q^2 \\ q & q_x & q_{xy} - q_{xx} \\ g & g_x & g_{xy} - g_{xx} \end{pmatrix}.$$

As $q \neq 0$ we get as second equation

$$q(g_x + g_{xx} - g_{xy}) - (q_x + q_{xx} - q_{xy})g - 2q_x(g_{xx} - g_{xy}) + 2(q_{xx} - q_{xy})g_x$$

and by subtracting a suitable multiple of the first equation it becomes divisible by $(\lambda^2 - 1)^2$, giving as final equations

$$\begin{aligned} &a^2 + 3ab + 3b^2 - 3a\lambda = 0\,, \\ &3(\lambda - 1)^2(b + a/3 - \lambda^2 + \lambda - 1)^2 + (\lambda + 1)^2(b + a + \lambda^2 - \lambda + 1)^2 = 0\,. \end{aligned}$$

These equations define a reducible curve, but no component is defined over $\mathbb{Q}$. To get a specific example note that in characteristic 31 there is a solution $\lambda = 2$, $a = 9$ and $b = -11$. For these values of the parameters one finds explicit points. A *Macaulay* computation shows that there is in fact a unique sextic with only isolated triple points in the ten points. This shows that for a general solution over $\mathbb{C}$ of the equations above a unique surface with ten triple points exists.

Computing the dimension of the tangent space to the equisingular stratum for the specific example gives $15+3$. However if one looks at the number of equations and the number of variables it seems that in the general case the solution space has to be one-dimensional.

If one leaves out P_7, P_8 or P_9, one finds a pencil of sextics with nine triple points in the remaining ones, which contains as reducible curve two conics with five points in it and a degree eight curve with one triple point and six nodes. This is a surface with nine triple points and $(c_1, c_2, c_3) = (2, 2, 8)$! So the sextic with ten triple points lies in the closure of several families. We get solutions in the closure of other families by applying Cremona transformations.

One could also start out with the family $\alpha q^3 + \beta q_1 q_2 q_3$. The surfaces with ten triple points in this family are reciprocally related with the surfaces in the other families with the same number of parameters. As the Cremona transformation depends on the position of the points it might be possible that one finds a real surface in this family. Unfortunately the corresponding equations are too difficult to solve.

Altogether we have proved the following

Theorem 4.16 *For every triple*

$$(c_1, c_2, c_3) \in \{(2, 2, 2), (2, 2, 4), (2, 2, 8), (2, 4, 6), (4, 4, 4)\},$$

the closure of the seven parameter family of sextics with $\nu = 9$ triple points and (-1)-curves of degrees c_1, c_2 and c_2 contains at least a one parameter family of rational sextics with ten triple points.

Corollary 4.17 $\mu_3(6) = 10$.

4.6 Summary

Theorem 4.18 *The sextics with triple points fall into 18 classes according to the following table.*

ν	c_1^2	c_2	χ	p_g	q	b_2	$h^{1,1}$	$\#(-1)$	κ	$\overline{X}$
0	24	108	11	10	0	106	86	0	2	general type
1	21	99	10	9	0	97	79	0	2	general type
2	18	90	9	8	0	88	72	0	2	general type
3	15	81	8	7	0	79	65	0	2	general type
4	12	72	7	6	0	70	58	0	2	general type
5	9	63	6	5	0	61	51	1	2	general type
5	9	63	6	5	0	61	51	0	2	general type
6	6	54	5	4	0	52	44	1	2	general type
6	6	54	5	4	0	52	44	0	2	general type
7	3	45	4	3	0	43	37	1	2	general type
7	3	45	4	3	0	43	37	0	2	general type
8	0	36	3	2	0	34	30	1	2	general type
8	0	36	3	2	0	34	30	2	2	general type
8	0	36	3	2	0	34	30	0	1	elliptic
8	0	36	3	3	1	36	32	0	1	elliptic
9	-3	27	2	1	0	25	23	3	1	elliptic
9	-3	27	2	1	0	25	23	3	0	$K3$
10	-6	18	1	0	0	16	16		$-\infty$	rational

All numbers denote invariants of the corresponding surface and $\#(-1)$ denotes the number of (-1)-curves distinguishing $\tilde{X}$ from its minimal model $\overline{X}$.

5 Higher degree

Surfaces of degree $d \geq 7$ with many triple points are surfaces of general type by corollary 3.4. It is, as with surface with many ordinary double points, very difficult to find explicit examples of high degree with many ordinary triple points. A septic surface ($d = 7$) can have at most 17 triple points by the spectrum bound.

We construct a one parameter family of septics with 16 triple points. The symmetric group S_4 acts on the polynomial ring $C[x, y, z, w]$ by permutation of the variables. The $\mathbb{C}$ vector space of all S_4-symmetric polynomials of degree seven has dimension eleven and is generated by the polynomials

$$\sigma_1^7,\ \sigma_1^5\sigma_2,\ \sigma_1^4\sigma_3,\ \sigma_1^3\sigma_2^2,\ \sigma_1^3\sigma_4,\ \sigma_1^2\sigma_2\sigma_3,\ \sigma_1\sigma_2^3,\ \sigma_1\sigma_2\sigma_4,\ \sigma_1\sigma_3^2,\ \sigma_2^2\sigma_3,\ \sigma_3\sigma_4\,.$$

Here σ_i denotes the i-th elementary symmetric polynomial, $i = 1, \ldots, 4$. Now take as 16 triple points the S_4-orbit of length four generated by $P_1 = (1:0:0:0)$

consisting of the vertices of the coordinate tetrahedron and an orbit of twelve points generated by a point $R_1 = (\lambda : \mu : \nu : \nu)$.

For a S_4-symmetric septic, the condition to have a triple point in P_1 implies that the coefficients of the first four polynomials vanish. Imposing a triple point in R_1 gives 10 equations in 7 coefficients, which by symmetry reduce to seven equations. This system of linear equations gives an 7×7 matrix whose determinant is up to a constant

$$\nu^5(\lambda-\mu)^4(\lambda-\nu)^5(\mu-\nu)^5(\lambda+\nu)(\mu+\nu)(\lambda+\mu+2\nu)^4 \\ \cdot (\lambda\mu-\nu^2)(2\lambda\mu+\lambda\nu+\mu\nu)(\lambda\mu+2\lambda\nu+2\mu\nu+\nu^2)^3.$$

It is easily checked that all solutions except $\lambda + \nu = 0$ or equivalently $\mu + \nu = 0$ correspond to either degenerate surfaces or to orbits with less than twelve points. For $\lambda + \nu = 0$ the orbit of triple points is generated by $(-\nu : \mu : \nu : \nu)$. The coefficients of the symmetric polynomials are now easily determined. For general values of $(\mu : \nu) \in \mathbb{P}^1$ one has indeed a septic with 16 isolated ordinary triple points. Computing the dimension of the tangent space to the equisingular stratum gives one, so all surfaces in the family have S_4-symmetry. There is no irreducible surface with 17 triple points in this family.

Theorem 5.1 *The general element of the one parameter family of S_4-symmetric septics given by*

$$(\mu-\nu)^3\nu(\sigma_1^2\sigma_2\sigma_3 - \sigma_1\sigma_3^2 - \sigma_1^3\sigma_4) - (\mu+\nu)\nu^3\sigma_1\sigma_2^3 - (\mu+\nu)(\mu^3-\nu^3)\sigma_2^2\sigma_3 \\ + (\mu+\nu)(\mu-\nu)^2(\mu+2\nu)\sigma_1\sigma_2\sigma_4 + (\mu+\nu)(\mu-\nu)^3\sigma_3\sigma_4 = 0$$

has 16 ordinary triple points as its only singularities.

Corollary 5.2 $16 \leq \mu_3(7) \leq 17$.

Remark 5.3 Whenever 16 points are invariant under the symmetric group S_4, it is tempting to ask if they form a Kummer configuration. This is not the case here.

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References

[AGV] *V.I. Arnol'd, S.M. Gusein-Zade, A.N. Varchenko, Singularities of Differentiable Maps, Volume II*, Birkhäuser, Boston (1988).

[BPV] *W. Barth, C. Peters, A. Van de Ven, Compact Complex Surfaces*, Springer, Berlin (1984).

[B–M] *D. Bayer, M. Stillman,* Macaulay: A system for computation in algebraic geometry and commutative algebra, Computer software available via anonymous ftp from:
`ftp://www.math.columbia.edu/pub/bayer/Macaulay/`

[C] *A. Cayley,* A memoir on quartic surfaces, In: Coll. Works **VII** (1894), 133–181.

[DV] *P. Du Val,* On isolated singularities of surfaces which do not affect the conditions of adjunction I–III, Proc. Camb. Philos. Soc. **30** (1934), 453–491.

[G] *D. Gallarati,* Sulle superficie del quinto ordine dotate di punti tripli, Rend. Accad. Naz. Lincei, serie **VIII,** vol. **XII** (1952), 70–75.

[M] *Y. Miyaoka,* The maximal number of quotient singularities on surfaces with given numerical invariants, Math. Annalen **268** (1984), 159–171.

[R] *K. Rohn,* Die Flächen vierter Ordnung hinsichtlich ihrer Knotenpunkte und ihrer Gestaltung, S. Hirzel (1886).

[SR] *J. G. Semple, L. Roth, Introduction to Algebraic Geometry*, Oxford University Press (1949).

[Y] *J.G. Yang,* On quintic surfaces of general type, Trans. Amer. Math. Soc. **295** (1986), 431–473.

[W] *J. Wall,* Miyaoka-Yau inequality for normal surfaces and local analogues, Contemp. Math. **162** (1994), 381–402.

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